oscode: fast solutions of oscillatory ODEs in cosmology

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Motivation

Algorithm

Applications
- Airy and 'burst' equations
- Quantum mechanics
- Cosmology

Extensions

Summary

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- Oscillators are extremely common in physics

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  2. Adaptive stepsize: update stepsize based on error estimate on step and tolerance.
- At each step, attempt to use both methods, and choose one which gives larger stepsize within the given error tolerance

Runge–Kutta and Wentzel–Kramers–Brillouin

RK

- $\dot{x} = F(x)$

[1] W. J. Handley, A. N. Lasenby, and M. P. Hobson. "The Runge-Kutta-Wentzel-Kramers-Brillouin Method". In: *arXiv e-prints* (Dec. 2016). arXiv: 1612.02288 [physics.comp-ph].

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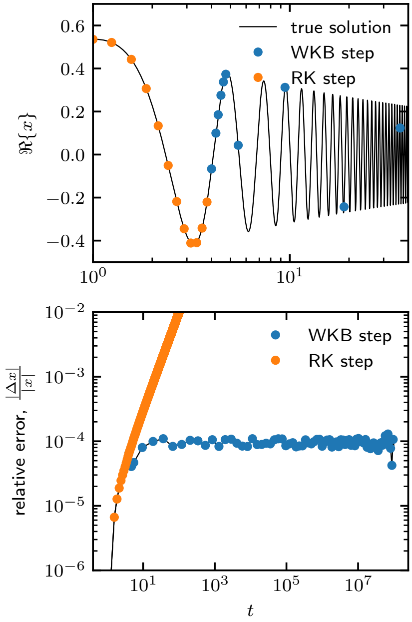
Airy equation

$\ddot{x} + tx = 0$

- Maximally hard for RK-based methods, there is analytic solution → study error properties



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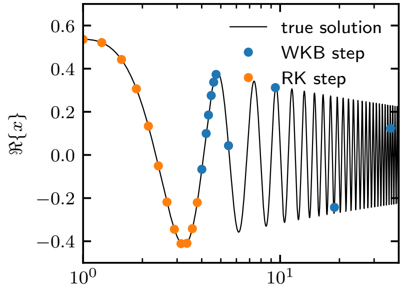

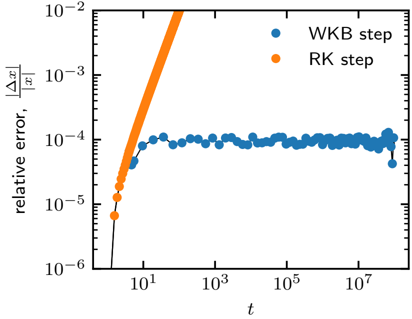

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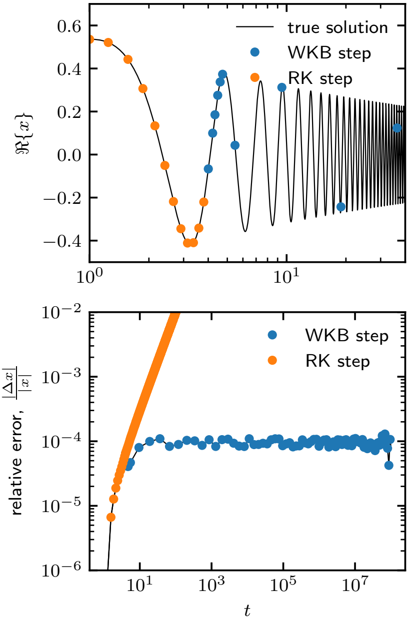
- `oscode` switches from RK to WKB early on, increases stepsize polynomially and stays within error tolerance (10^{-4})



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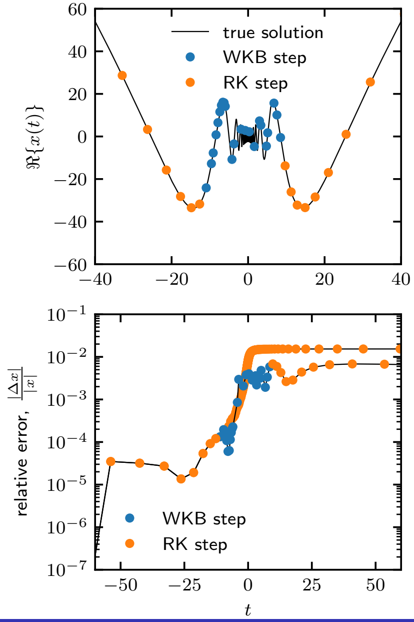
$\ddot{x} + \frac{n^2-1}{(1+t^2)^2}x = 0$



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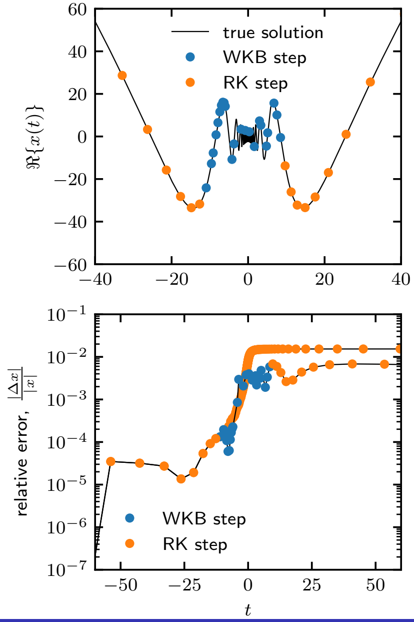

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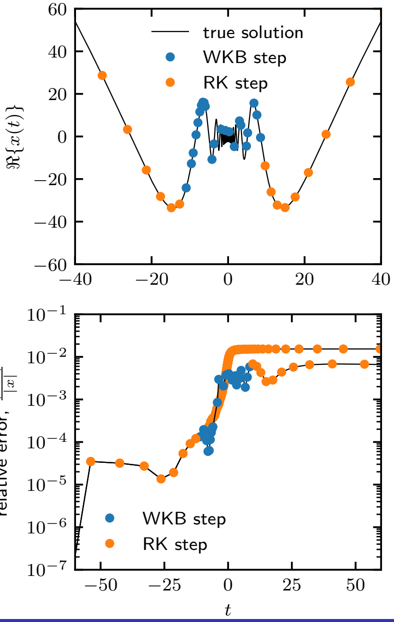

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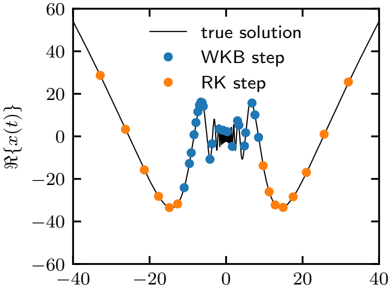

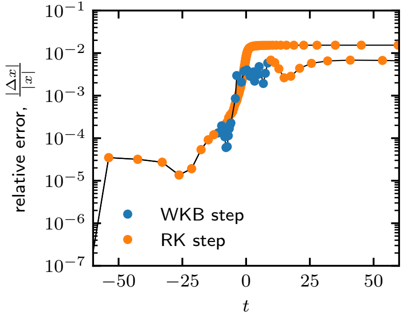

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- $\sim$ symmetric switching

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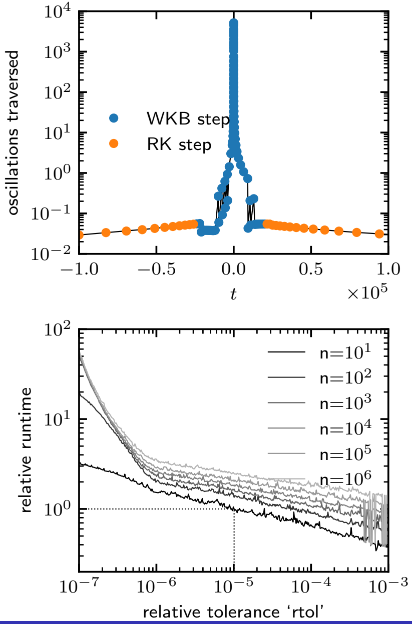
- $n = 10^5$ pictured (but can go up to 10^{10}, then true solution is hard to plot due to sine and cosine's accuracy)



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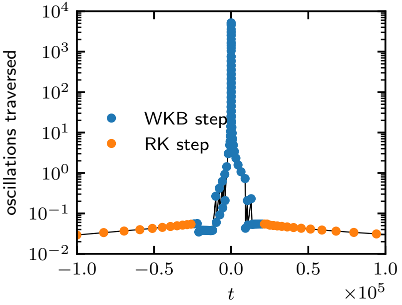

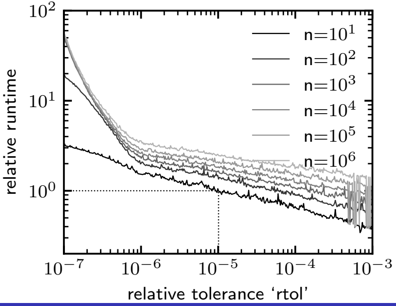

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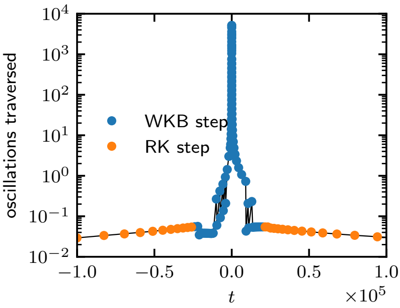

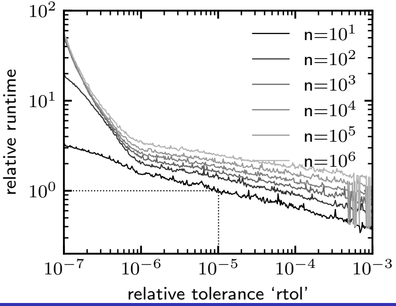

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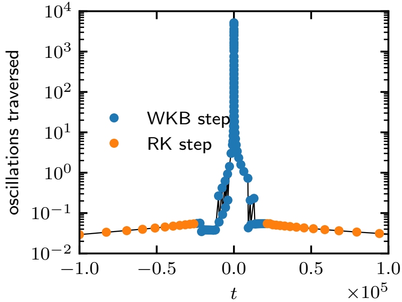
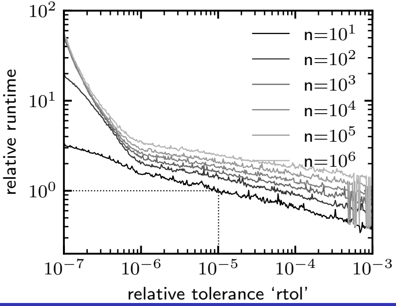
- Gentle scaling of runtime within $10^{-6} <$tol$< 10^{-4}$





Schrödinger equation

$\Psi''(x) + 2m(E - V(x))\Psi(x) = 0$

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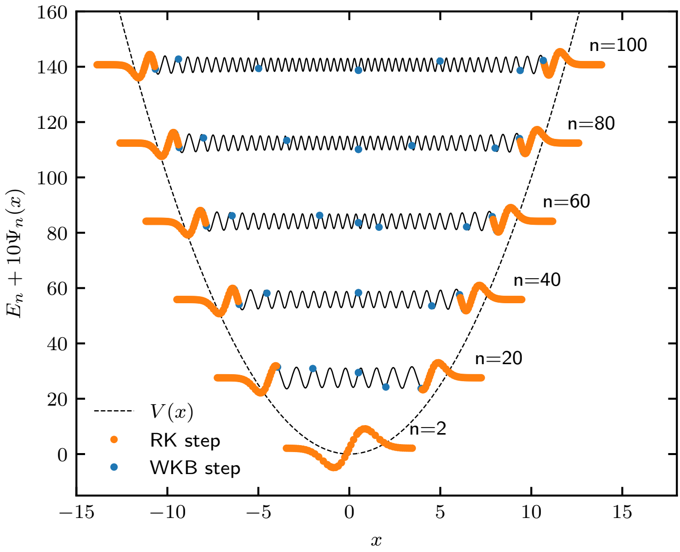
- Eigenvalues obtained match reality much more closely than the tolerance set

Harmonic potential well

$V(x) = x^2$



Harmonic well + quartic anharmonicity

$V(x) = x^2 + \lambda x^4$, $\lambda = 1$

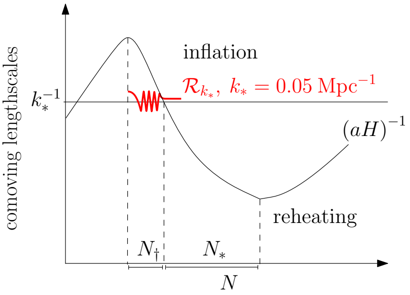
n	E_n^{oscode}	E_n^{*}[2]	$\sim \log_{10} \lvert\Delta E/E\rvert$
0	1.392353	1.392352	-6
1	4.648815	4.648813	-7
2	8.6550501	8.6550500	-8
3	13.156806	13.156804	-7
4	18.0577	18.0576	-5
15	88.6104	88.6103	-6
16	96.1291	96.1296	-5
17	103.793	103.795	-5
18	111.6025	111.6020	-6
19	119.5440	119.5442	-6
50	417.05620	417.05626	-7
100	1035.5440	1035.5442	-7
1000	21932.7848	21932.7840	-8
10000	471103.81	471103.80	-8

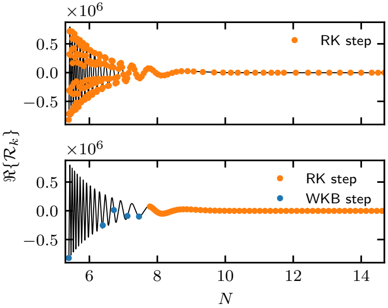
[2] K. Banerjee et al. "The Anharmonic Oscillator". In: *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences* 360.1703 (1978).

Mukhanov–Sasaki equation

$$\ddot{\mathcal{R}}_k + 2\left(\frac{\ddot{\phi}}{\dot{\phi}} - \frac{1}{2}\dot{\phi}^2 + \frac{3}{2}\right)\dot{\mathcal{R}}_k + \left(\frac{k}{aH}\right)^2 \mathcal{R}_k = 0$$

- Governs evolution of curvature perturbation $\mathcal{R}_k$ with lengthscale k^{-1}





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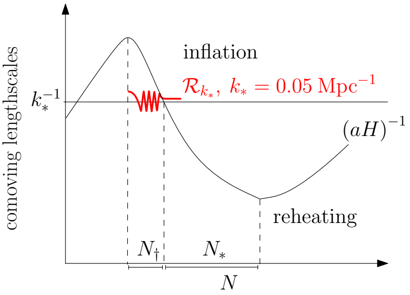

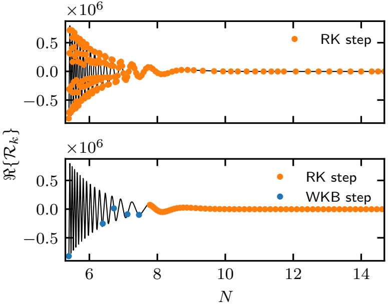

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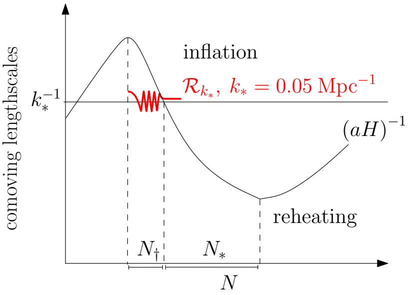

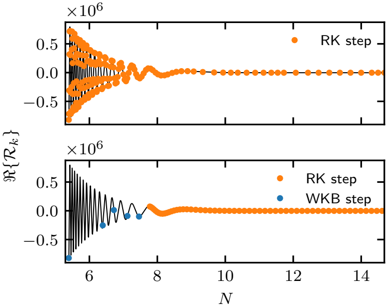

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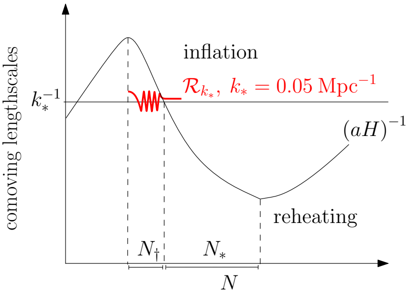
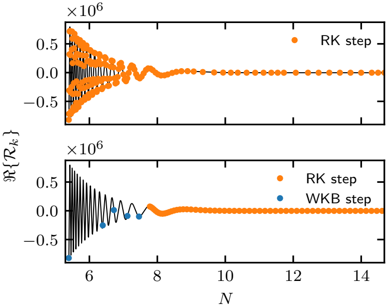
- Power spectrum of $\mathcal{R}_k$ is the primordial power spectrum (PPS), precursor of the CMB





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[3] L. T. Hergt et al. "Case for kinetically dominated initial conditions for inflation". In: *Phys. Rev. D* 100 (2 July 2019), p. 023502.

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[5] W. I. J. Haddadin and W. J. Handley. *Rapid numerical solutions for the Mukhanov-Sazaki equation*. 2018. arXiv: 1809.11095 [astro-ph.CO].

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- But this is challenging at large k

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$$\ddot{\mathcal{R}}_k + 2\left(\frac{\ddot{\phi}}{\dot{\phi}} - \frac{1}{2}\dot{\phi}^2 + \frac{3}{2}\right)\dot{\mathcal{R}}_k + \left(\frac{k}{aH}\right)^2 \mathcal{R}_k = 0$$

- Need to compute PPS numerically for many inflationary models, e.g. kinetic dominance[3][4]
- But this is challenging at large k
- Other fast solvers exist, but rely on assumptions[5]

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[3] L. T. Hergt et al. "Case for kinetically dominated initial conditions for inflation". In: *Phys. Rev. D* 100 (2 July 2019), p. 023502.

[4] L. T. Hergt et al. "Constraining the kinetically dominated universe". In: *Phys. Rev. D* 100 (2 July 2019), p. 023501.

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- Need to compute PPS numerically for many inflationary models, e.g. kinetic dominance[3][4]
- But this is challenging at large k
- Other fast solvers exist, but rely on assumptions[5]
- Speed up forward-modelling phase of inference significantly (> 1000x), e.g. closed-universe models[6]

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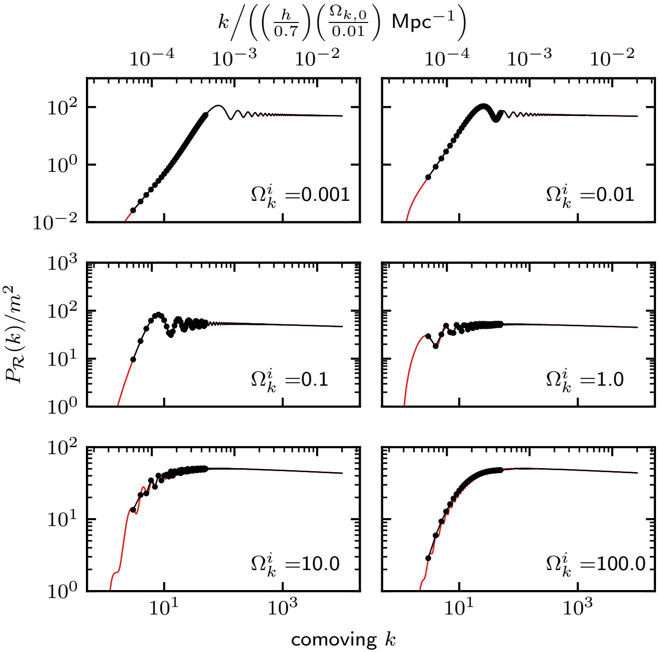
Closed universes



Extensions

- Generalising to many dimensions (is challenging)[7]

[7] Jamie Bamber and Will Handley. "Beyond the Runge-Kutta-Wentzel-Kramers-Brillouin method". In: *arXiv e-prints* (July 2019). arXiv: 1907.11638 [physics.comp-ph].

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Extensions

- Generalising to many dimensions (is challenging)[7]
- Generalising to higher order ODEs
- Use an approximation other than WKB
- `oscode` and its underlying algorithm are the beginning of a novel suite of methods

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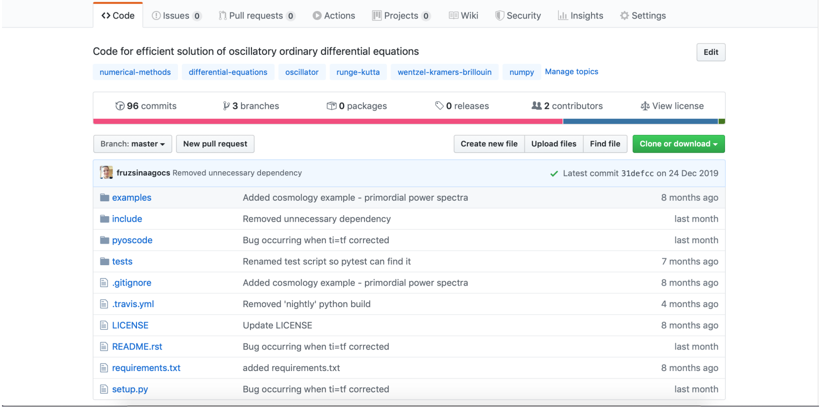
Open-source software, documentation, examples



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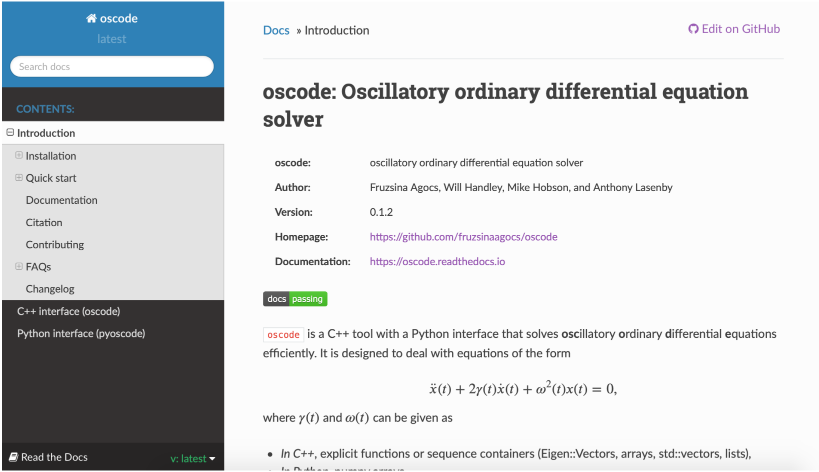
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Changelog
C++ interface (oscode)
Python interface (pyoscode)
Read the Docs v: latest

Docs » Introduction
Edit on GitHub

oscode: Oscillatory ordinary differential equation solver

oscode:	oscillatory ordinary differential equation solver
Author:	Fruzsina Agocs, Will Handley, Mike Hobson, and Anthony Lasenby
Version:	0.1.2
Homepage:	https://github.com/fruzsinaagocs/oscode
Documentation:	https://oscode.readthedocs.io

docs passing

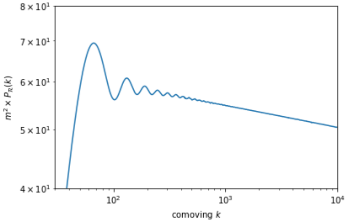
`oscode` is a C++ tool with a Python interface that solves **osc**illatory **o**rdinary **d**ifferential **e**quations efficiently. It is designed to deal with equations of the form

$$\ddot{x}(t) + 2\gamma(t)\dot{x}(t) + \omega^2(t)x(t) = 0,$$

where $\gamma(t)$ and $\omega(t)$ can be given as

- *In C++*, explicit functions or sequence containers (Eigen::Vectors, arrays, std::vectors, lists),

Open-source software, documentation, examples



A closed universe

All we have to do differently is:

1. solve the background equations again with $K = 1$,

In [37]:
```
K = 1
N_i = -1.74
ok_i = 1.0
N = np.linspace(N_i,N_f,Nbg)
# Initial conditions
phi_i = np.sqrt(4.*(1./ok_i + K)*np.exp(-2.0*N_i)/m**2)
logok_i = np.log(ok_i)
y_i = np.array([logok_i, phi_i])
# Solve for the background until the end of inflation
endinfl.terminal = True
endinfl.direction = 1
bgsol = solve_ivp(bgeqs, (N_i,N_f), y_i, events=endinfl, t_eval=N, rtol=1
e-8, atol=1e-10)
```

Number of e-folds of inflation now is

Summary

- `oscode` is a numerical solver for oscillatory ordinary differential equations[8]

[8] F. J. Agocs et al. "Efficient method for solving highly oscillatory ordinary differential equations with applications to physical systems". In: *Phys. Rev. Research* 2 (1 Jan. 2020), p. 013030.

Summary

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Summary

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- Underlying algorithm switches between methods depending on whether solution is oscillatory
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- Wide range of uses: quantum mechanics, electrical circuits, cosmology, ...

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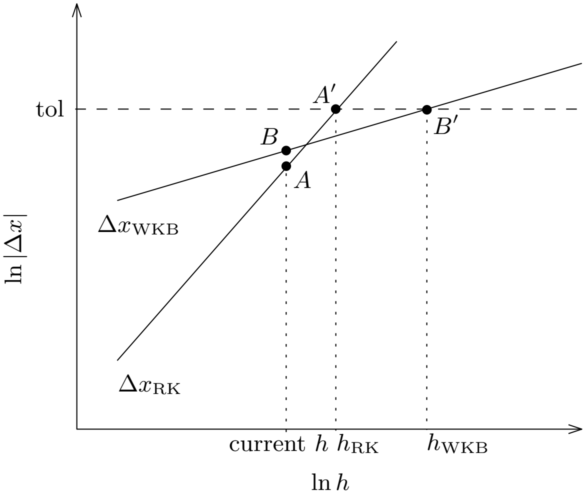
Error estimates



Gauss–Lobatto integration

Gaussian quadrature

$$\int_a^b w(x)f(x)dx \simeq \sum_{i=1}^{n} w_i f(x_i)$$

Gauss – Lobatto quadrature

$$\begin{array}{ll} interval(a,b): & [-1,\ 1] \\ w(x): & 1 \\ polynomials: & P'_{n-1}(x) \end{array}$$

Gauss – Lobatto quadrature

$$\int_{-1}^{1} f(x)dx \simeq \frac{2}{n(n-1)}(f(-1)+f(1)) + \sum_{i=2}^{n-1} w_i f(x_i)$$

$$\int_a^b f(x)dx \simeq \frac{b-a}{2}\left[\frac{2(f(a)+f(b))}{n(n-1)} + \sum_{i=1}^{n} w_i f\left(\frac{b-a}{2}x_i + \frac{b+a}{2}\right)\right]$$

Extended WKB

$$\ddot{x} + 2\gamma\dot{x} + T^2\omega^2 x = 0. \tag{1}$$

$$x(t) \sim \exp\left(T\sum_{n=0}^{\infty} S_n(t)T^{-n}\right). \tag{2}$$

$$\dot{S}_0(t) = \pm i\omega, \tag{3}$$

$$\dot{S}_i(t) = -\frac{1}{2S_0'}\left(\ddot{S}_{i-1} + 2\gamma\dot{S}_{i-1} + \sum_{j=1}^{i-1}\dot{S}_j\dot{S}_{i-j}\right). \tag{4}$$