Reproducing the unique acoustics of periodic staircases with boundary integral equations

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0. Summary

Corrugated, periodic surfaces that reflect sound (are "sound-hard") can give rise to trapped acoustic modes that travel along the surface and are evanescent away from it. We reproduce these trapped modes numerically, find their dispersion relation, and model scattering from a nearby quasiperiodic array of point sources, with an aim to simulate a single point source by using the array scanning method. We combine the trapped mode dispersion with a simple ray model to explain the chirp-like impulse response observed at long, periodic structures such at the El Castillo below [1].

1. Geometry and problem setup

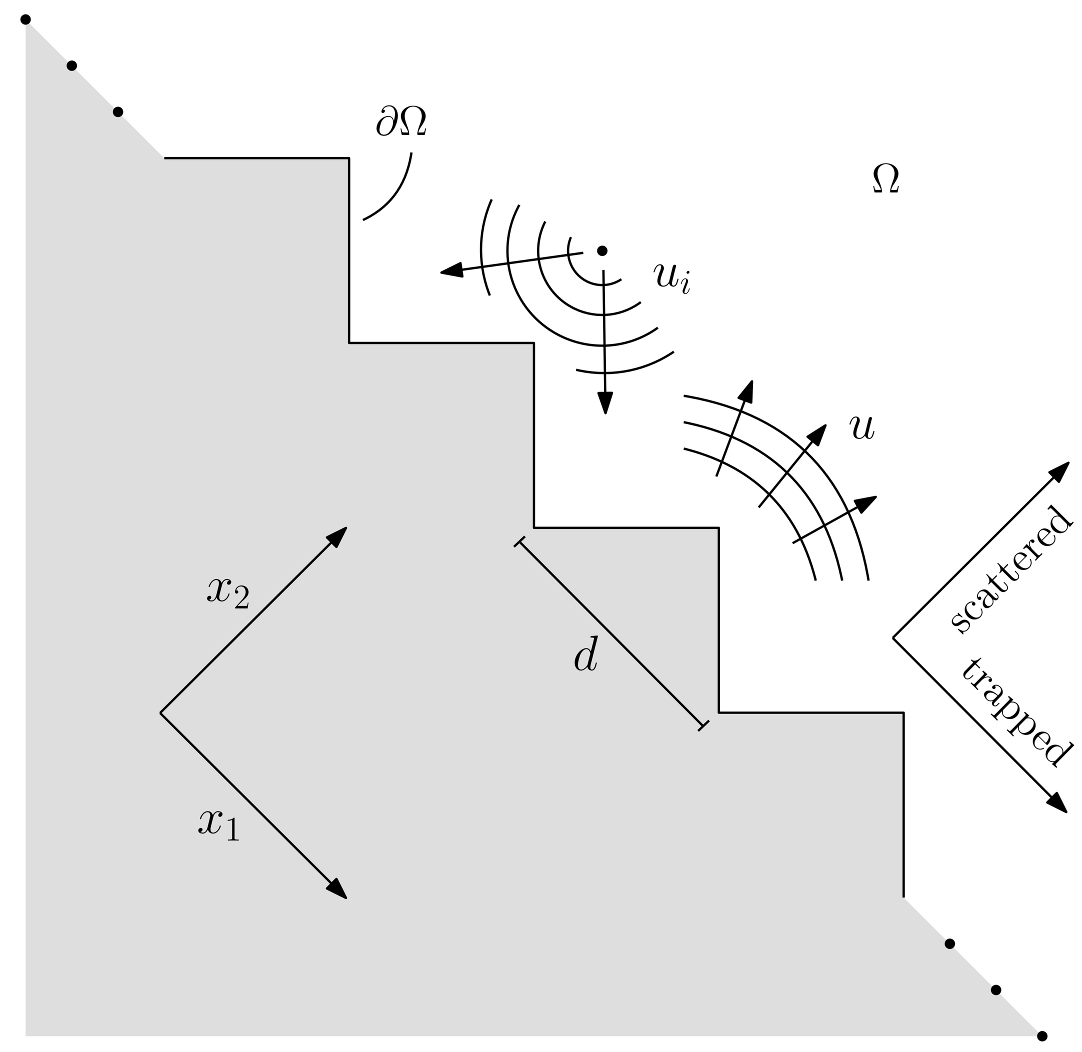

Fig. 1. Geometry of the problem. u_i is an incoming wave, u the scattered wave. The coordinates are rotated to be parallel and normal to the slope of the stairs, and the unit cell (a single step) is of width d.

Fig. 1. Shows the geometry of the problem. It is posed as a Neumann BVP with an upwards propagating radiation condition. The quasiperiodicity condition ensures that the solution obeys the symmetry of the boundary, accruing a **Bloch phase** of $\alpha = e^{i\kappa}$ over a period, where κ is the **horizontal (on-surface) wavenumber.**

$(\Delta + \omega^2)u = 0$	in Ω,	PDE
$u_n = 0$	on $\partial\Omega$,	boundary condition
$u(x_1 + nd, x_2) = \alpha^n u(x_1, x_2)$	$(x_1, x_2) \in \Omega$,	quasiperiodicity
$u(x_1, x_2) = \sum_{n\in\mathbb{Z}} c_n e^{i\kappa_n x_1 + k_n x_2}$,	$x_2 > x_2^{(0)}$	radiation condition

If the total wavevector is $\vec{k} = (\kappa_n, k_n)$, with the set of possible horizontal wavenumbers $\kappa_n = \kappa + 2\pi n,\ n \in \mathbb{Z}$, all corresponding to the same Bloch phase, then $k_n = \sqrt{\omega^2 - \kappa_n^2}$.

2. Methods

- *Periodization:* We reduce the computation to one unit cell [2] by using the periodic Green's function (GF) $\Phi_p(\vec{x}, \vec{y})$,

$$-(\Delta + \omega^2)\Phi_p(\vec{x}, 0) = \delta(x_2)\sum_{n=-\infty}^{\infty} \alpha^n \delta(x_1 - nd).$$

We separate the GF into near- ($|n| \leq 1$) and far-field parts, and write the far-field contribution as a **Neumann series**

$$\Phi_{p,\text{far}}(\vec{x}, 0) = \frac{i}{4}\left[S_0(\omega,\kappa)J_0(\omega,\vec{x}) + 2\sum_{n=1}^{\infty} S_n(\omega,\kappa)J_n(\omega,\vec{x})a(\vec{x})\right],$$

where the J_n are the Bessel functions, and the S_n are referred to as **lattice sums**. The S_n involve sums over nth order Hankel functions and are thus are slowly convergent, but may be **represented with integrals** [3] which are readily computed using numerical quadrature. Since they are independent of the target location ($\vec{x}$), they only need to be computed once per wavenumber.

- *Boundary integral:* We use a single-layer representation for the scattered wave,

$$u = \mathcal{S}\sigma = \int_{\partial\Omega} \Phi_p(x, y)\sigma(y)\mathrm{d}s_y.$$

Using the appropriate jump relations yields the Fredholm integral equation

$$(I - 2D^{\mathrm{T}})\sigma = -2f,$$

Where the boundary data is $f = -(u_i)_n|_{\partial\Omega}$, D is the double-layer kernel, and σ the unknown density. We solve the integral equation via Nystrom's method, which involves numerical quadrature on nodes chosen carefully on the boundary to avoid the **corner singularities**.

- *Corner refinement:* Initially, the boundary is split into equally sized panels with an equal number of Gauss—Legendre quadrature nodes on each, then corner-adjacent panels are split in a $1:(r-1)$ ratio, getting smaller towards the corner. The node coordinates are relative to the nearest corner, to avoid loss of accuracy due to catastrophic cancellation.

- *Finding trapped modes:* Trapped modes are eigenmodes of the Neumann BVP; they occur when there is a non-trivial solution to

$$(I - 2D^{\mathrm{T}})\sigma = 0,$$

i.e. when $\det(I - 2D^{\mathrm{T}}) = 0$. We find them by fixing κ, then using Newton's method to find the roots of the Fredholm determinant as a function of ω.

- *Array scanning:* Due to use of the periodic GF, non-periodic wave sources are not allowed. Scattering from a single point source, however, may be computed using the following trick [4]. If a periodic array of point sources with Bloch phase κ is written

$$\mathcal{J}_p(\vec{x}, \kappa) = \sum_{n=-\infty}^{\infty} \delta(x_1 - nd)\delta(x_2)e^{in\kappa},$$

then a single point source is

$$\mathcal{J}(\vec{x}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathcal{J}_p(\vec{x}, \kappa)\mathrm{d}\kappa.$$

The scattered wave from a point source can thus be obtained by integrating along κ in the first **Brillouin zone**, $\kappa = [-\pi, \pi]$.

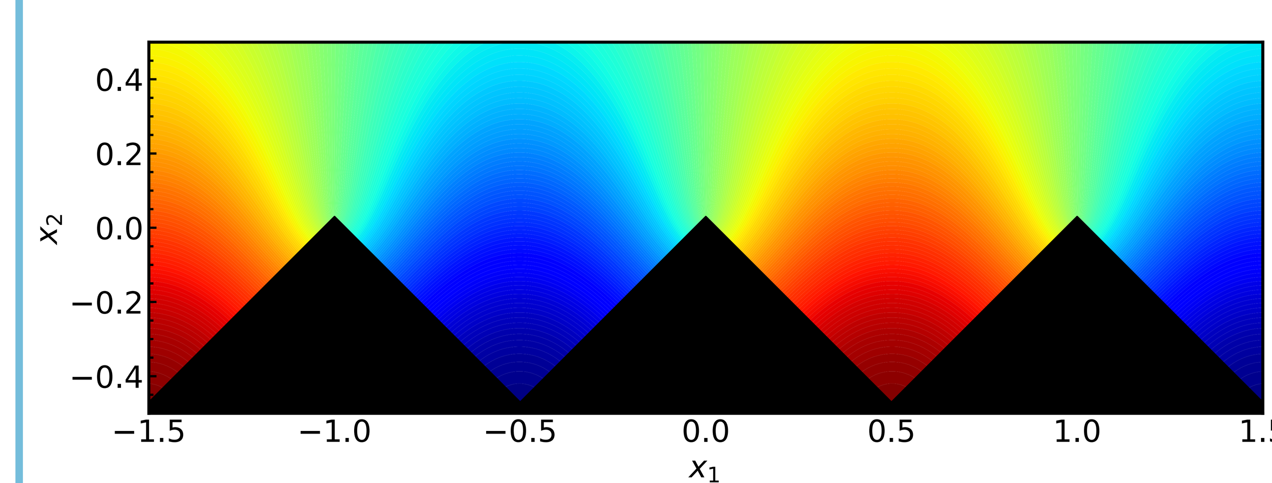

Fig. 3. Trapped surface wave at $\kappa = \pi$, the edge of the first Brillouin zone. This is the most trapped mode, i.e. with thhe shortest vertical decay length.

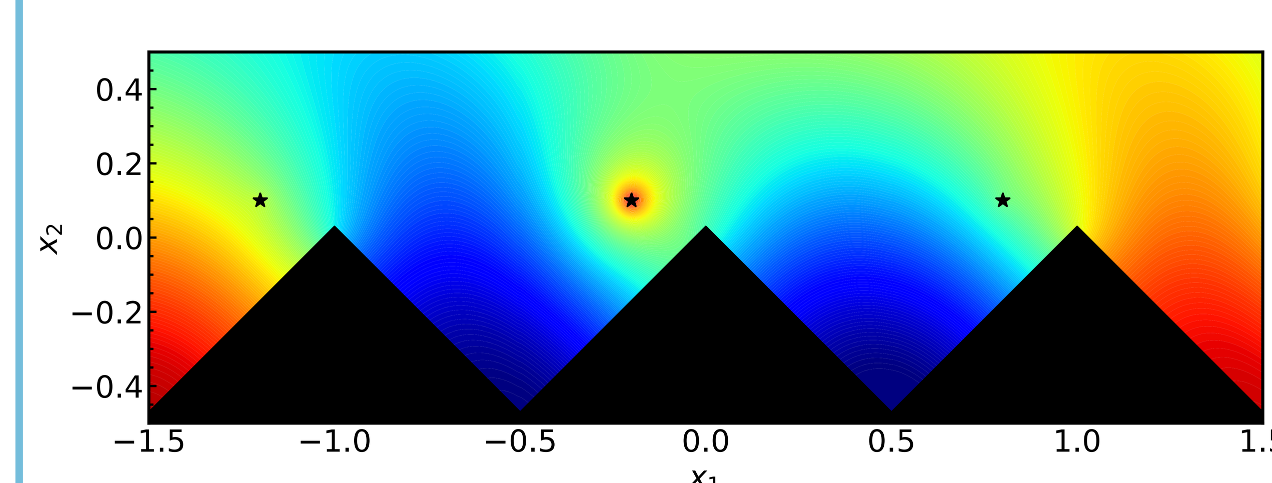

Fig. 6. Total field from an array of point sources, their positions marked by stars. We set $\omega = 3$, $\kappa = 1.5$.

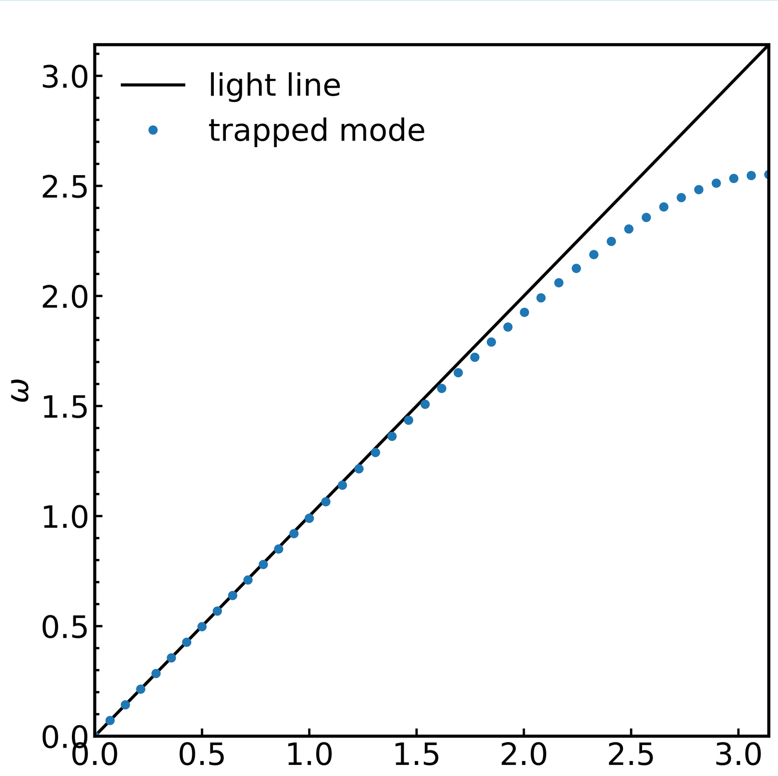

Fig. 2. Dispersion relation, frequency ω against horizontal wavenumber κ, for trapped modes within the first Brillouin zone.

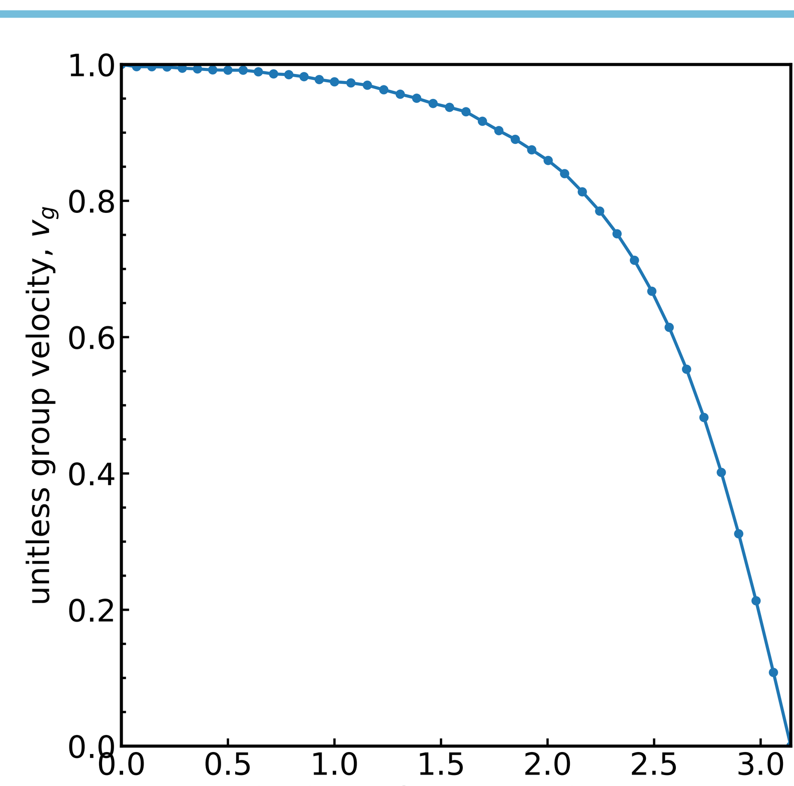

Fig. 4. Group velocity, $v_g = \mathrm{d}\omega/\mathrm{d}\kappa$, computed from the dispersion relation, as a function of horizontal wavenumber κ.

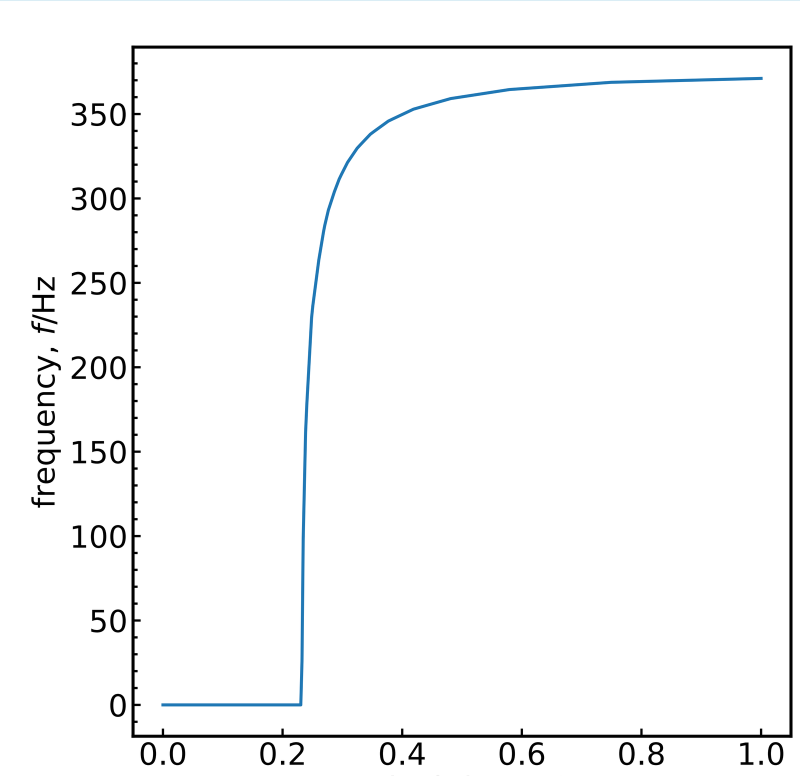

Fig. 5. Arrival time of modes of given frequencies at a given distance along a staircase modeled after El Castillo.

3. Results

- *Dispersion relation of trapped modes:* **Fig. 2.** shows the dispersion relation $\omega(\kappa)$ found for the first Brillouin zone. With Neumann boundary conditions, there is always a trapped mode, but as $\kappa \to 0$, it approaches the **light line**, $\omega = \kappa$, becoming increasingly weakly trapped (with a longer upwards decay length). The strongest trapped mode, at $\kappa = \pi$, is shown in **Fig. 3;** no trapped modes exist above the associated frequency ω.

- *Group velocity:* Dispersion is the separation of modes during wave propagation due to the difference of **phase and group velocities**. A mode of horizontal wavenumber κ travels at $v_g = \mathrm{d}\omega/\mathrm{d}\kappa$. A plot of $v_g(\kappa)$ in **Fig. 4.** shows this separation. By inverting $\omega(\kappa)$ from **Fig. 2.**, we compute $v_g(\omega)$, and from that the times at which different frequency components from a wavepacket arrive at a given distance along the stairs. **Fig. 5.** shows an example, computed using parameters of the stairs of El Castillo at Chichen Itza. Converted to audio, this would sound like a sharp rise (a "chirp"), followed by long ringing.

- *Scattering from an array of point sources:* **Fig. 6.** shows the total field from an array of point sources, with ω, κ chosen to be far from a Wood anomaly. The treatment of Wood anomalies in array scanning is the subject of ongoing work.

4. Future and ongoing work

- Frequency- and time-domain solution from nearby point source with array scanning:
  - Wood anomalies on the light line makes it necessary to (1) deform the complex contour, or (2) use a special quadrature rule.
- Push to high-order accuracy and fast solution time
  - Special quadrature rules for nearby (non-colinear) panel interactions.
  - Faster evaluation of periodic GF via local-to-local expansion.
- Analyze power in trapped modes: does the trapping get exponentially weaker as $\kappa \to 0$?

References

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