An adaptive spectral method for oscillatory second-order linear ODEs with frequency-independent cost

Fruzsina Agocs[1] with Alex Barnett[1]

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ICIAM Tokyo, August 2023

arxiv:2212.06924 (accepted to SINUM)

Acknowledgements

Thanks to:

			
Alex Barnett	Manas Rachh	Jim Bremer	Charlie Epstein

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- Interested in solving the initial value problem

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  - Is arbitrarily high-order.

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- Automatic **switching** between the methods

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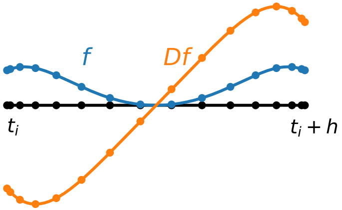
$$\left[\mathrm{D}^2 + \mathrm{diag}(\omega^2(\mathbf{t}))\right]\mathbf{u} = 0$$



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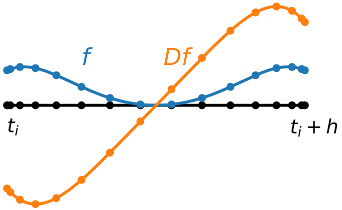

- To this $n \times n$ system, add two rows encoding initial conditions:

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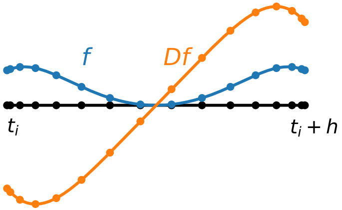

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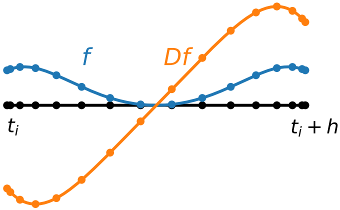

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- Get error estimate from repeating the step with $2n$ Chebyshev points and comparing $u_n(t_{i+1})$ with $u_{2n}(t_{i+1})$. Typically, $n = 16$.

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The nonoscillatory phase function

- Rewrite $u'' + \omega^2 u = 0$ [3] using $u = e^z$, and $z'(t) = x(t)$:

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- Algorithm is complex and only works if $\omega(t)$ is large

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$$x_1 = i\omega - \frac{\omega'}{2\omega}, \qquad R[x_1] = -\frac{\omega''}{2\omega} + \frac{3(\omega')^2}{4\omega^2} = \mathcal{O}(1).$$

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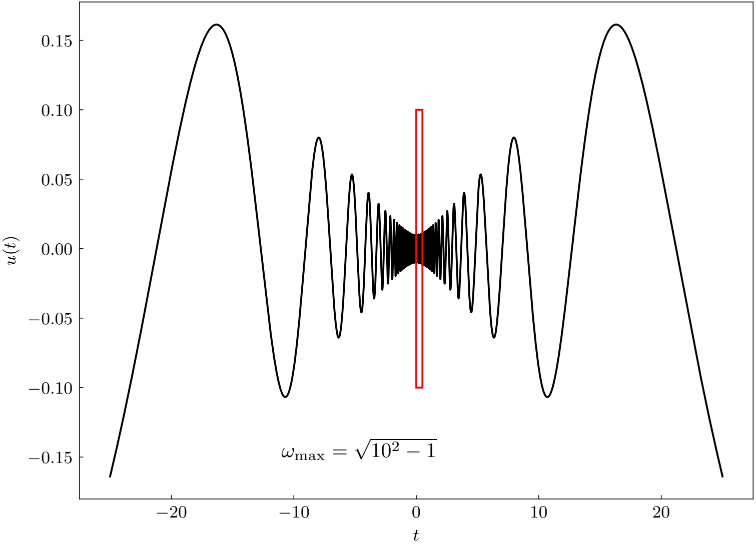

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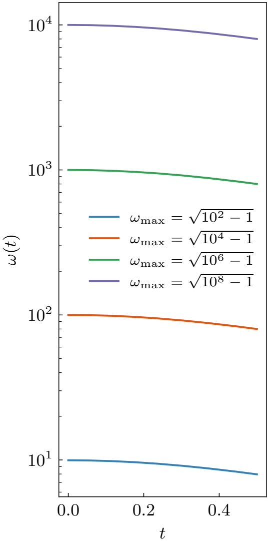

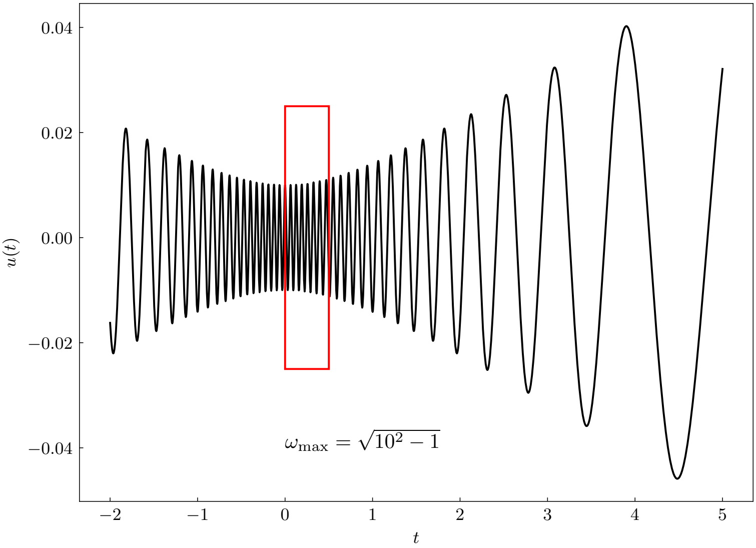

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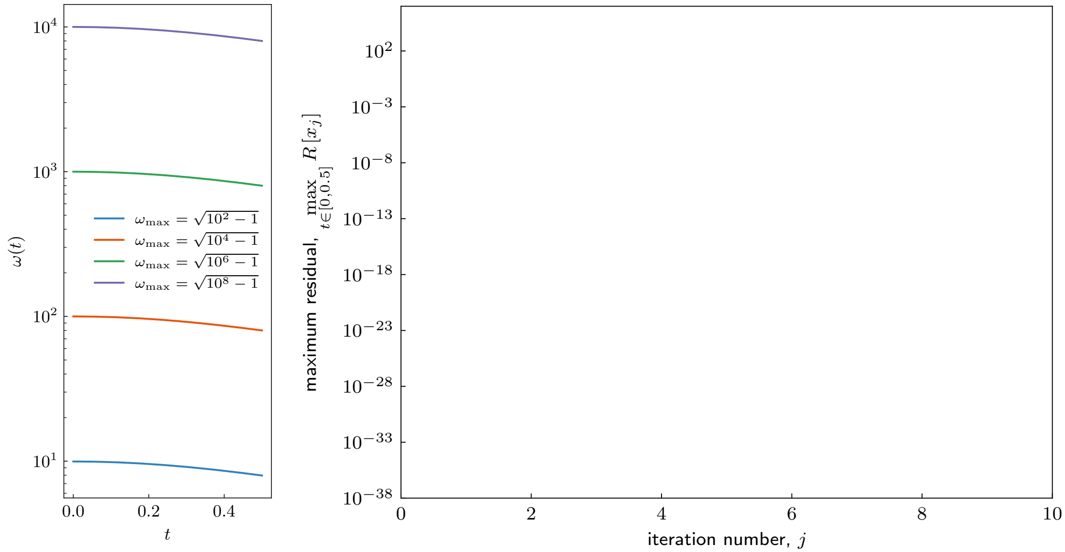

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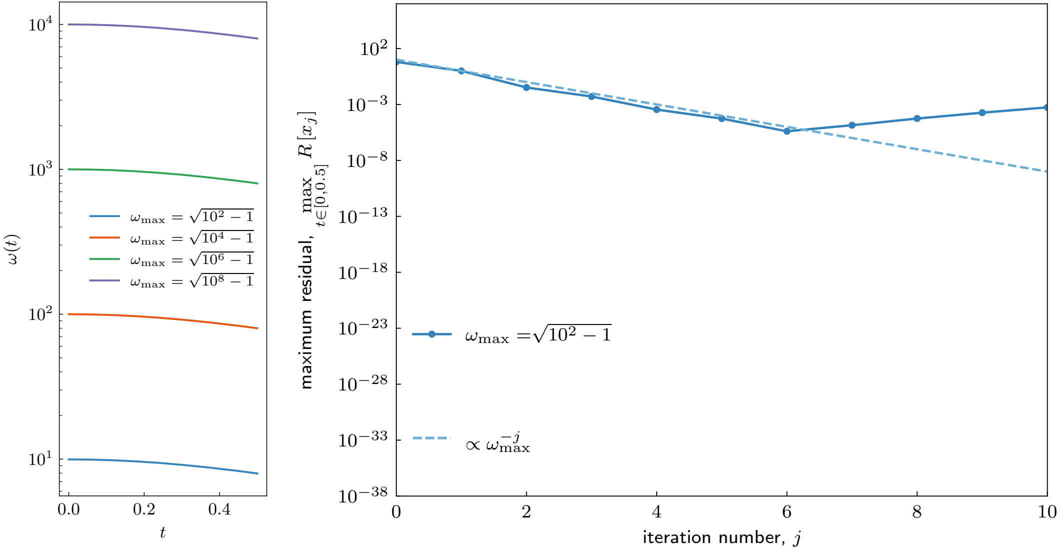

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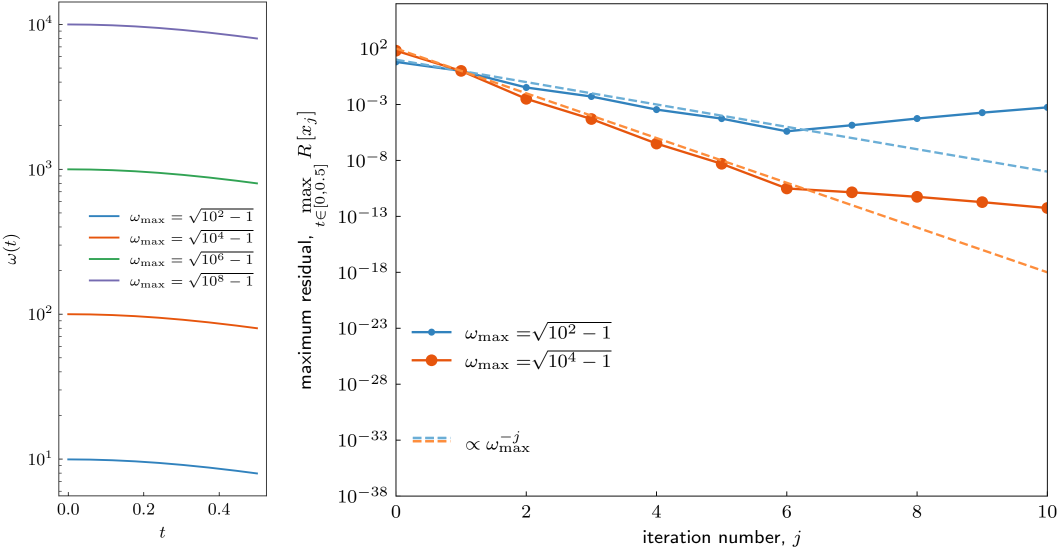

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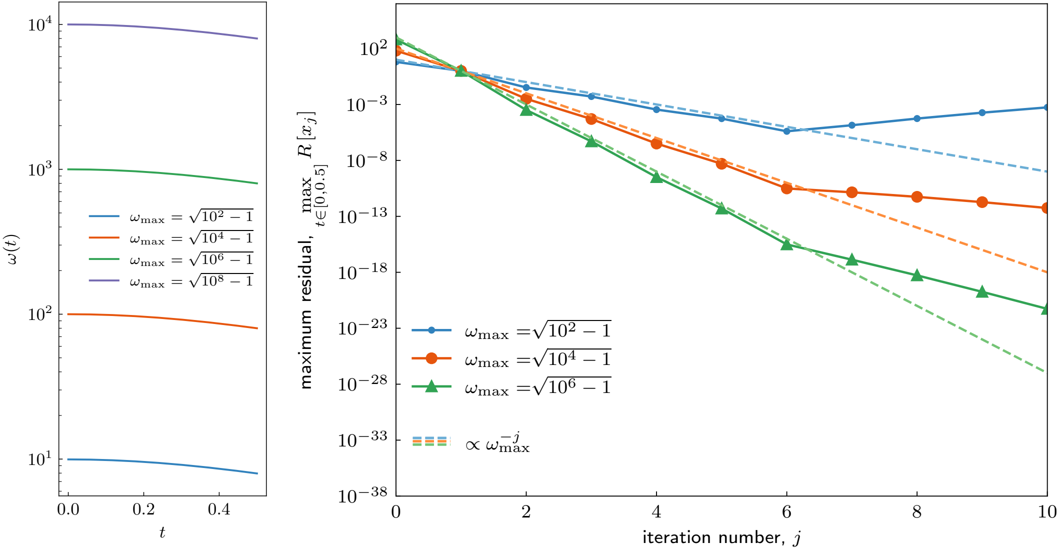

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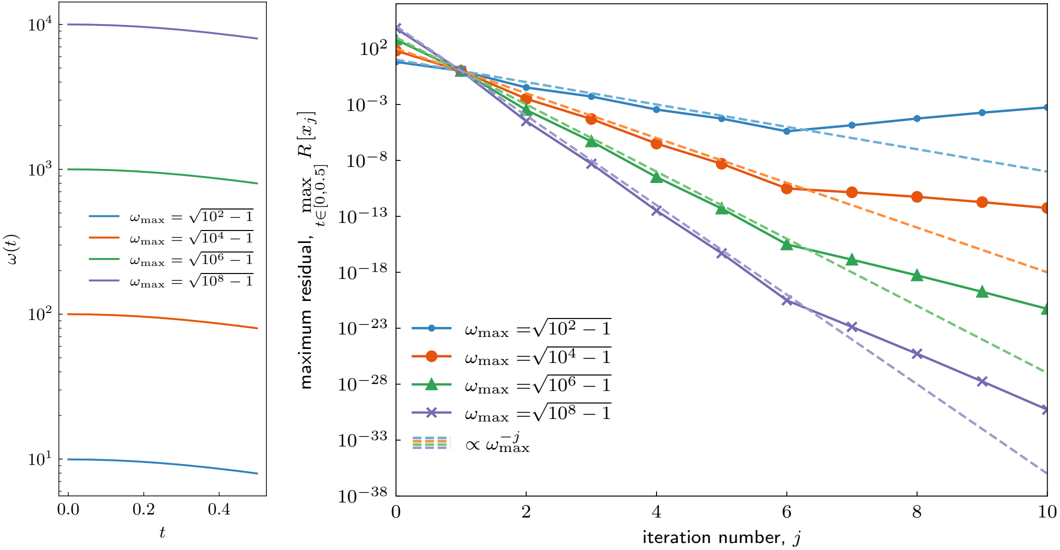

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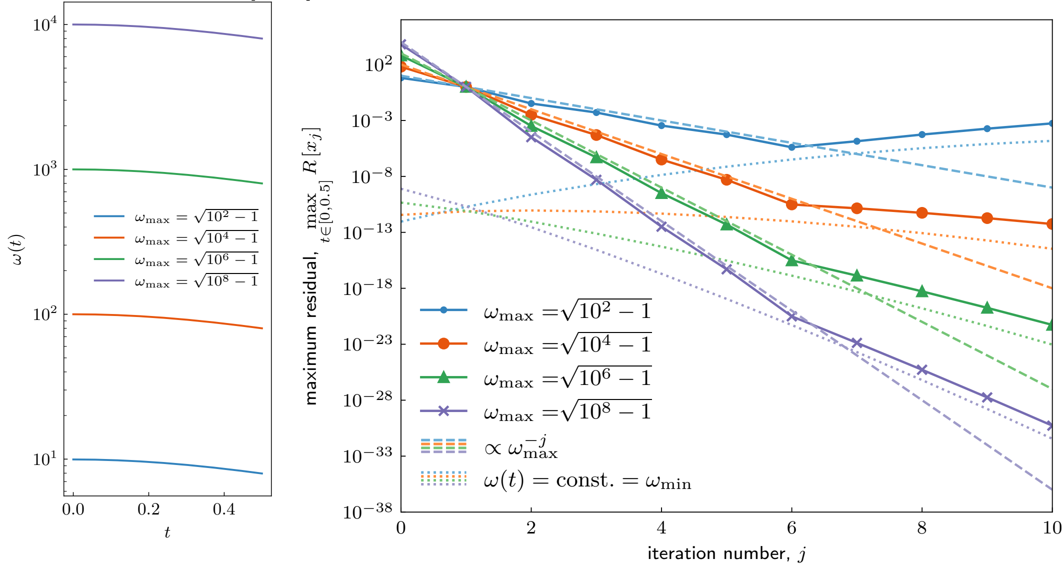

Geometric convergence of the residual, $R[x_j]$, *for a while*: a theorem

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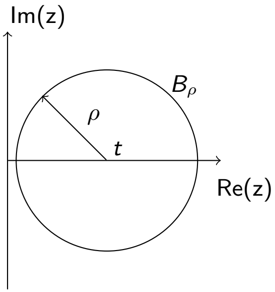

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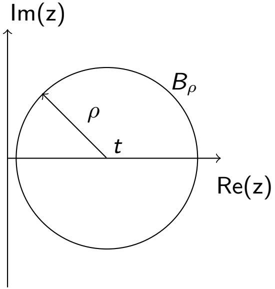

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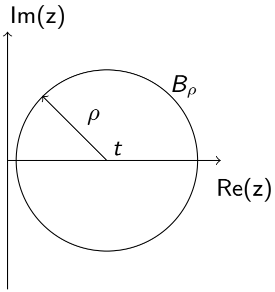

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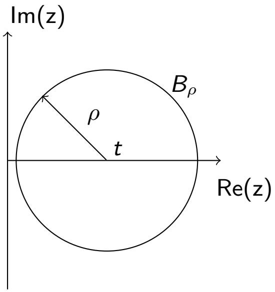
- Note: The theorem generalises to the $\gamma(t) \ne 0$ case by introducing an upper bound on γ.



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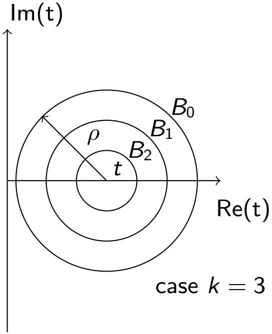

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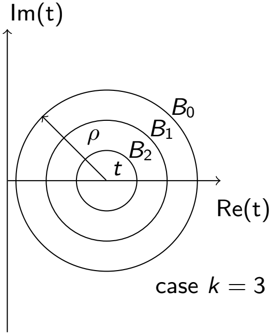

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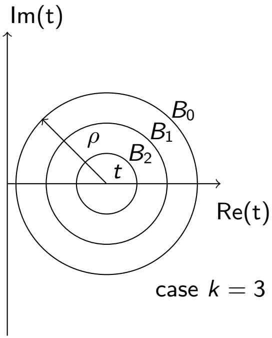
- Prove by induction that for iteration j,

$$\tilde{\eta}_1 \leq |x_l| \leq \tilde{\eta}_2 \quad \text{in } B_j, \quad \text{for all } l = 0, 1, \ldots, j,$$
$$|R_l| \leq \eta_3 r^l \quad \text{in } B_j, \quad \text{for all } l = 0, 1, \ldots, j.$$



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- Derivatives and integral via spectral differentiation / integration matrix ($n = 16, 32$) $\rightarrow$ stepsize determined only by how well ω, γ are represented on a Chebyshev grid

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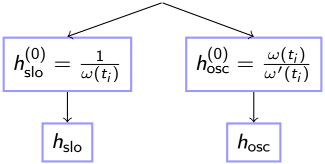
1. Get initial stepsize estimate

Algorithm overview



In stepping from t_i to $t_{i+1} = t_i + h$:

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2. Refine stepsize estimate

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In stepping from t_i to $t_{i+1} = t_i + h$:

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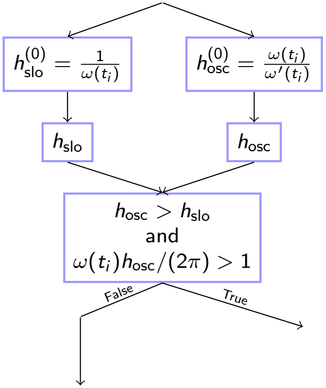

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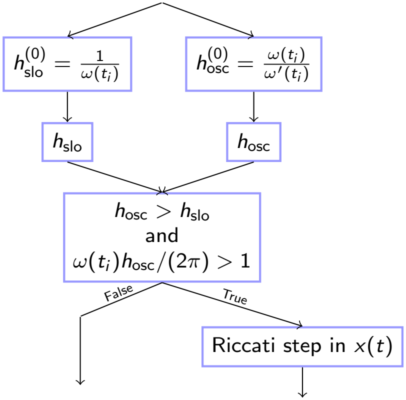

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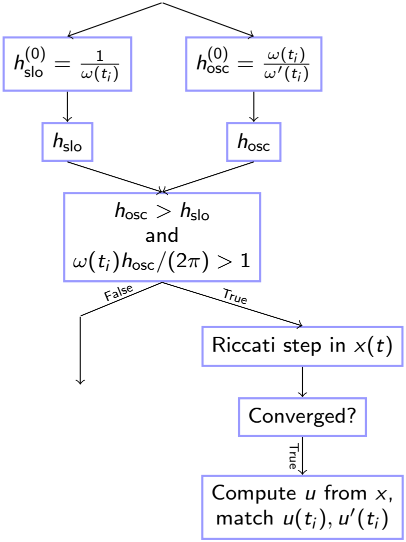

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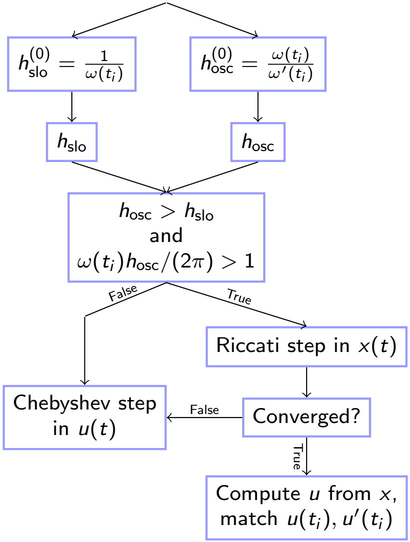

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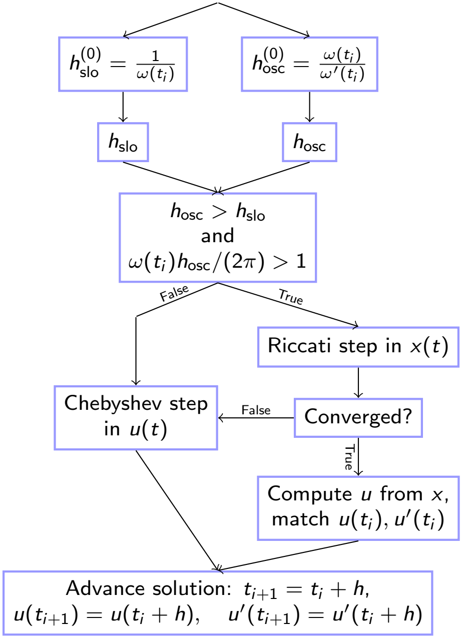

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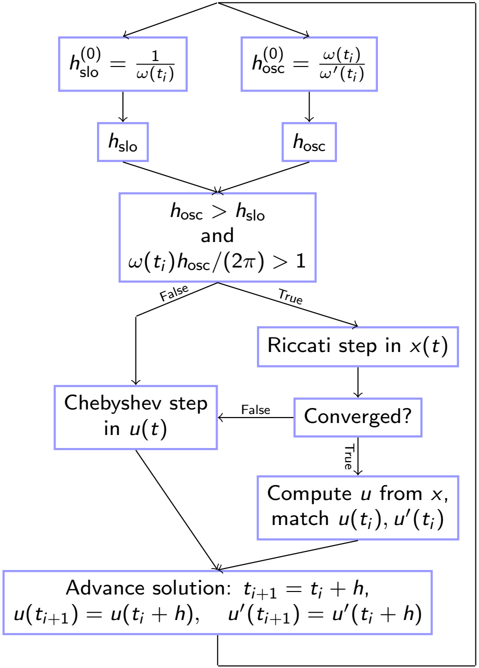

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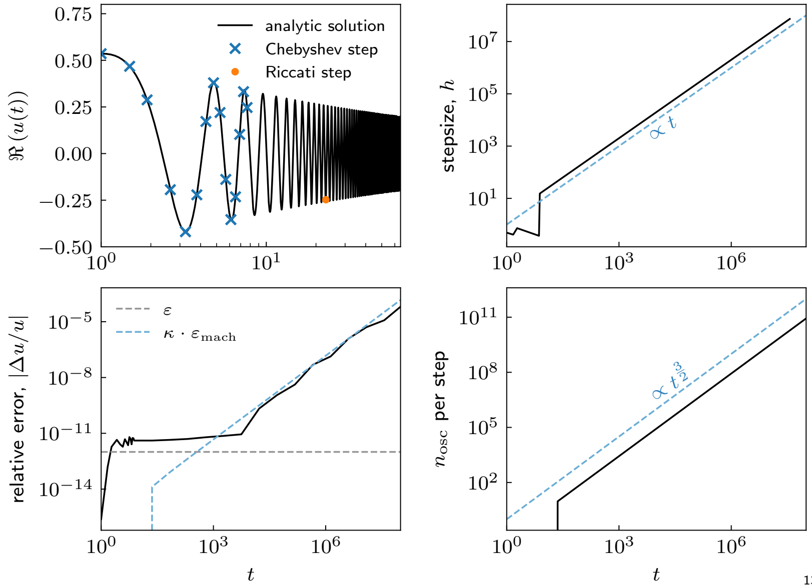

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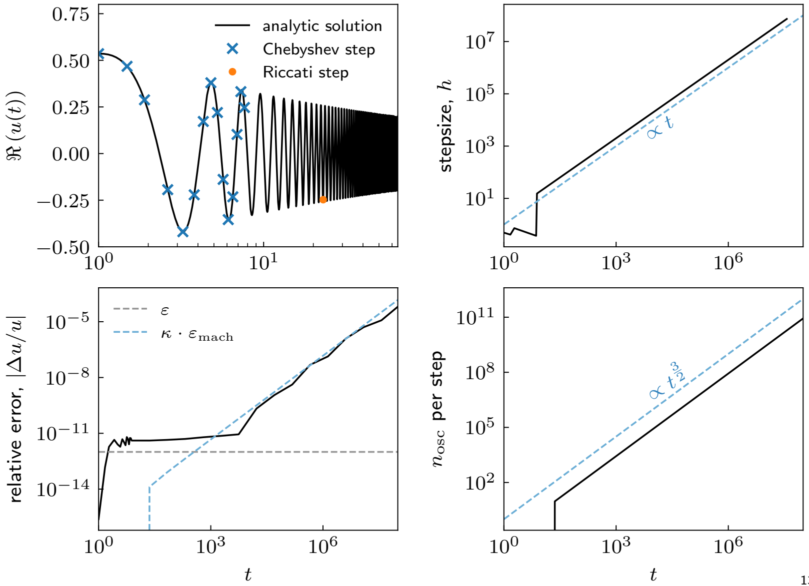

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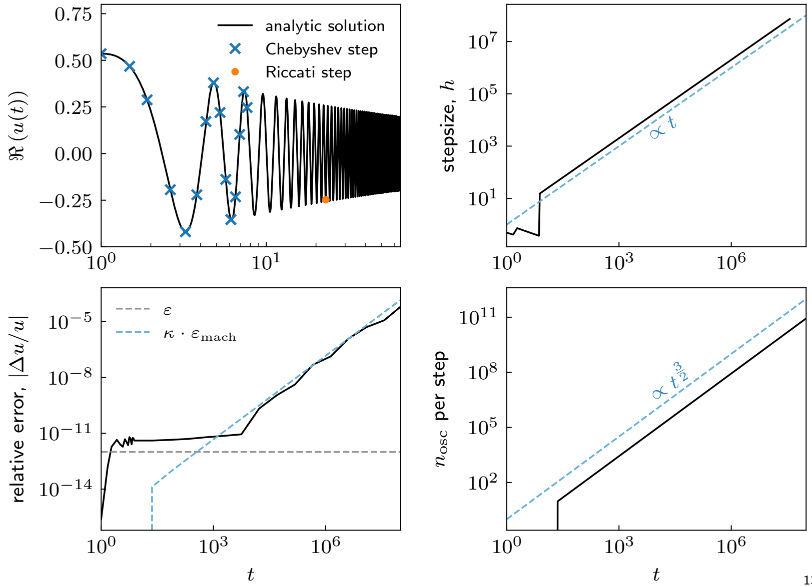

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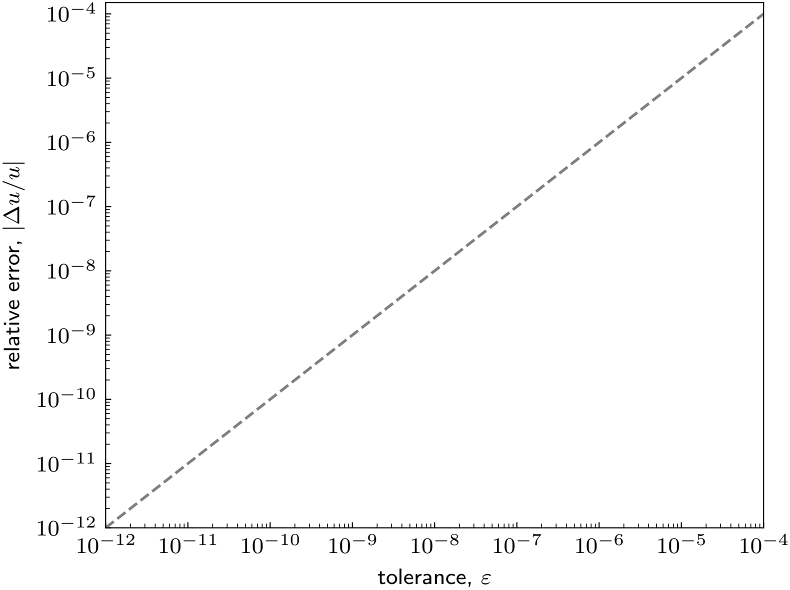

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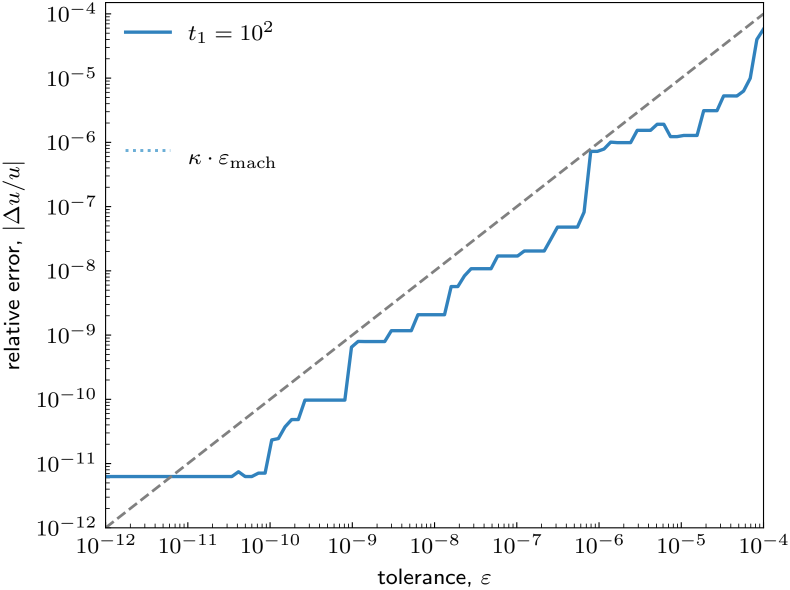

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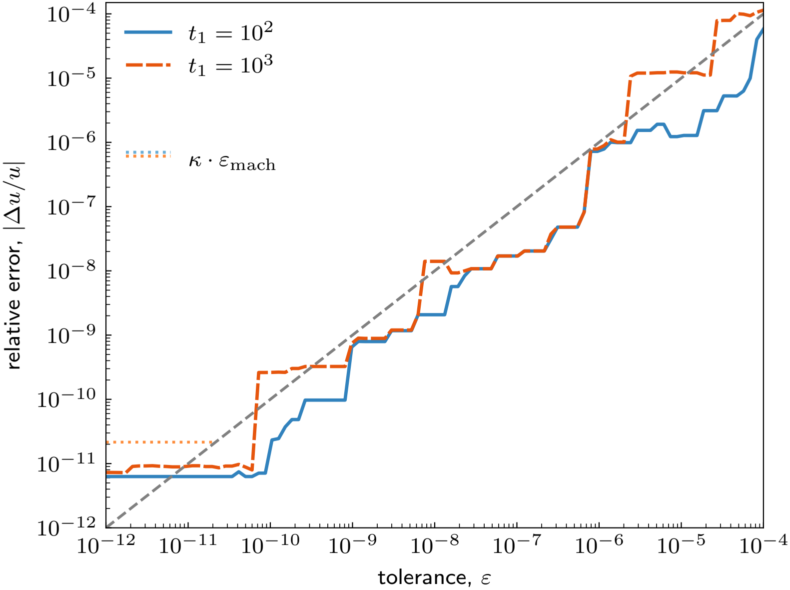

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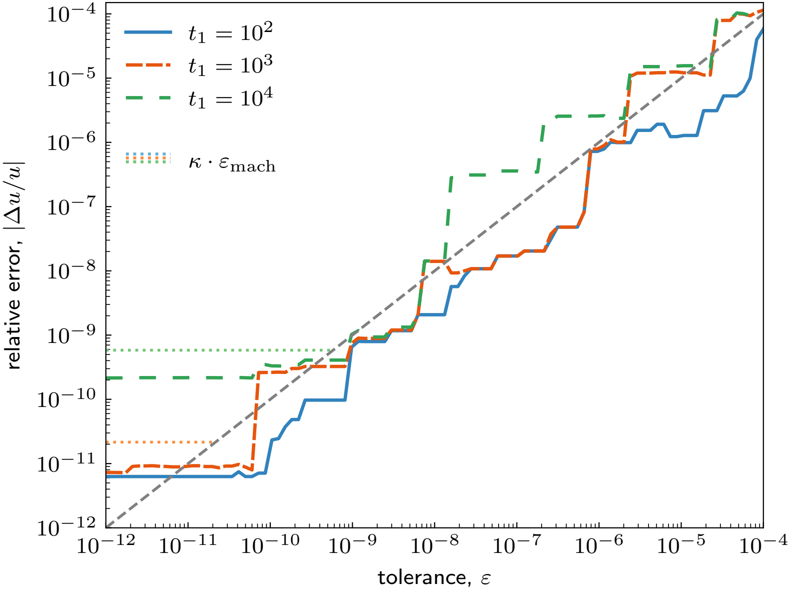

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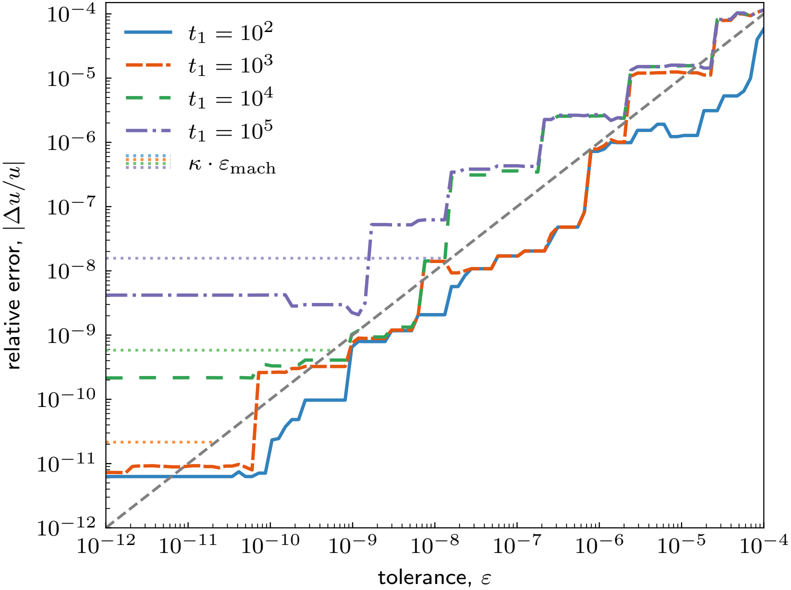

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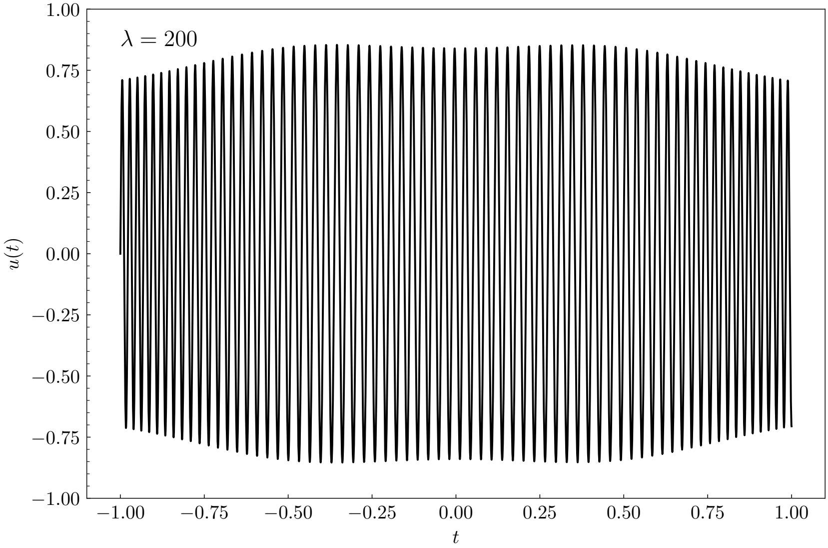

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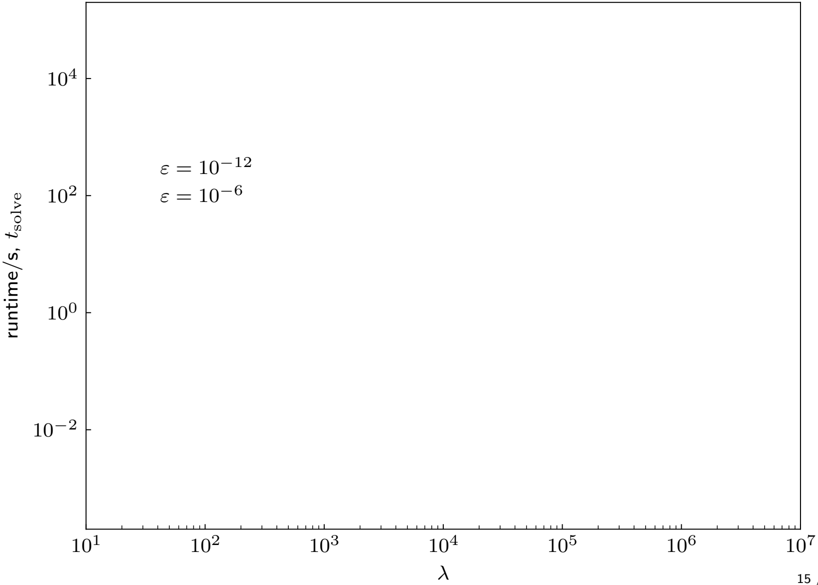

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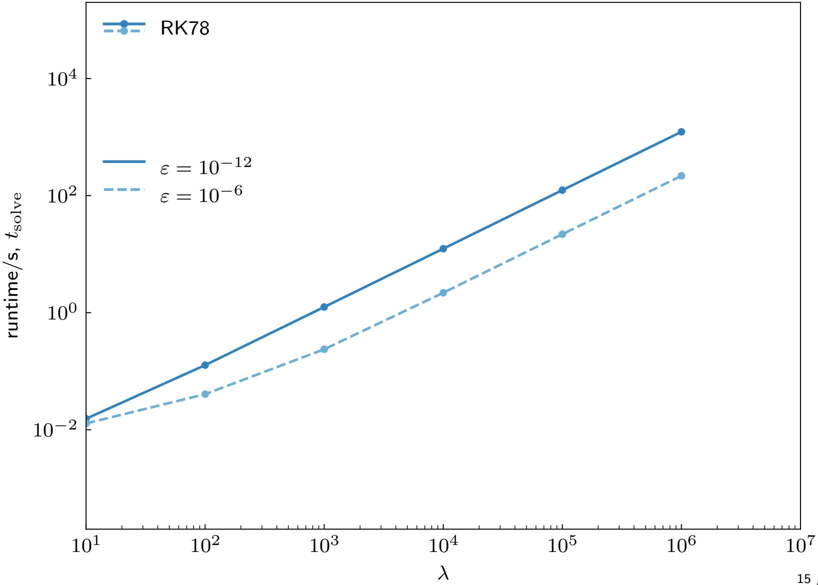

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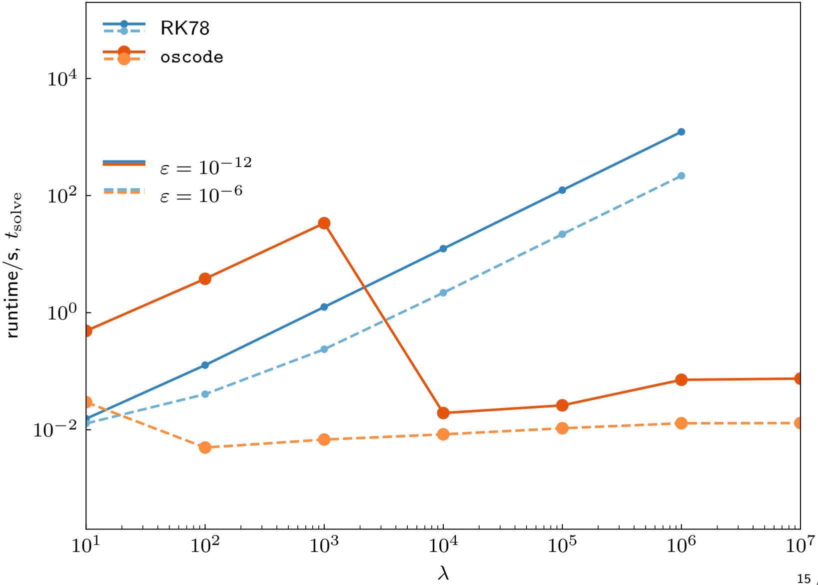

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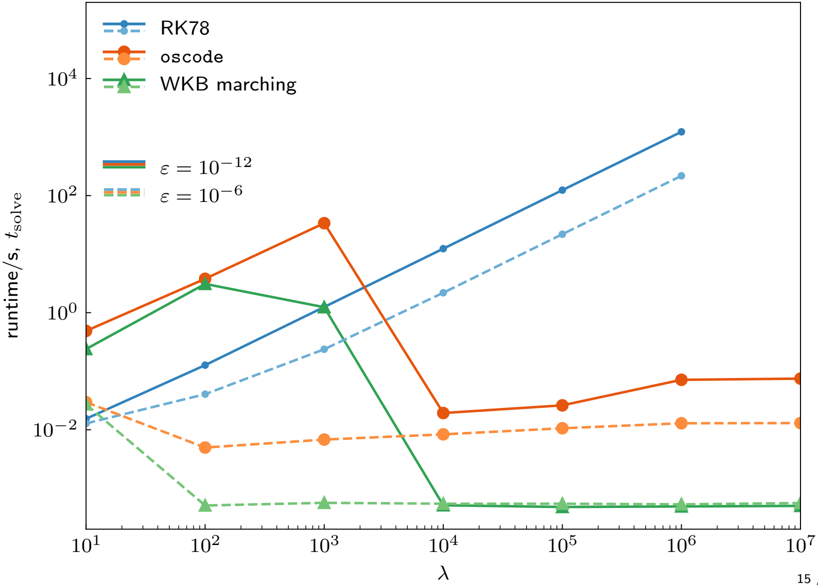

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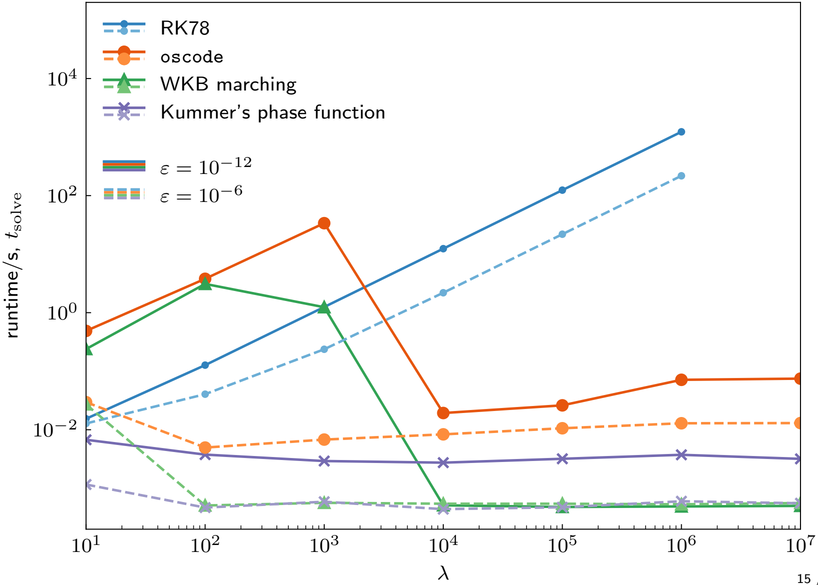

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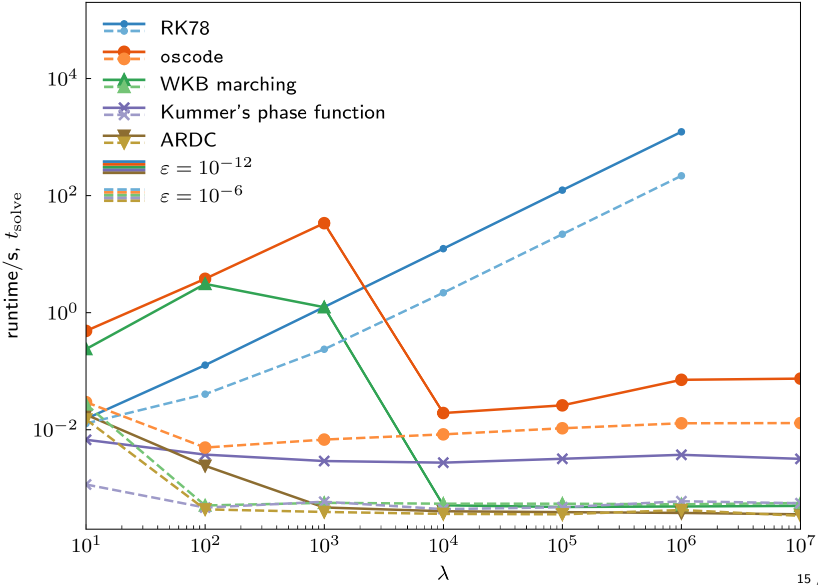

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Riccati

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- Unique: asymptotic methods applied numerically, spectral accuracy, can deal with oscillatory or slowly-varying regions, works in presence of friction term
- Asymptotic expansions reduce the residual **very quickly**, up until a certain iteration/term
- Could we generalise the method to ODE systems? PDEs?

Thank you!

References I

F. J. **Agocs** and A. H. Barnett (2022). *An adaptive spectral method for oscillatory second-order linear ODEs with frequency-independent cost*. DOI: 10.48550/ARXIV.2212.06924. URL: https://arxiv.org/abs/2212.06924.

F. J. **Agocs**, W. J. Handley, A. N. Lasenby, and M. P. Hobson (2020). "Efficient method for solving highly oscillatory ordinary differential equations with applications to physical systems". In: *Phys Rev Research* 2.1, p. 013030.

F. J. **Agocs**, L. T. Hergt, W. J. Handley, A. N. Lasenby, and M. P. Hobson (2020). "Quantum initial conditions for inflation and canonical invariance". In: *Phys Rev D* 102.2. ISSN: 2470-0029. DOI: 10.1103/physrevd.102.023507. URL: http://dx.doi.org/10.1103/physrevd.102.023507.

J. Bremer (2018). "On the numerical solution of second order ordinary differential equations in the high-frequency regime". In: *ACHA* 44.2, pp. 312–349.

— (2023). "Phase function methods for second order linear ordinary differential equations with turning points". In: *ACHA* 65, pp. 137–169. ISSN: 1063-5203. DOI: https://doi.org/10.1016/j.acha.2023.02.005. URL: https://www.sciencedirect.com/science/article/pii/S1063520323000210.

Z. Heitman, J. Bremer, and V. Rokhlin (2015). "On the existence of nonoscillatory phase functions for second order ordinary differential equations in the high-frequency regime". In: *JCP* 290, pp. 1–27.

References II

- L. T. Hergt, F. J. **Agocs**, W. J. Handley, M. P. Hobson, and A. N. Lasenby (2022). "Finite inflation in curved space". In: *Phys Rev D* 106.6. ISSN: 2470-0029. DOI: 10.1103/physrevd.106.063529. URL: http://dx.doi.org/10.1103/physrevd.106.063529.
- J. Körner, A. Arnold, and K. Döpfner (2022). "WKB-based scheme with adaptive step size control for the Schrödinger equation in the highly oscillatory regime". In: *JCAM* 404, p. 113905.
- M. I. Letey, Z. Shumaylov, F. J. **Agocs**, W. J. Handley, M. P. Hobson, and A. N. Lasenby (2022). *Quantum Initial Conditions for Curved Inflating Universes*. arXiv: 2211.17248 [gr-qc].
- L. R. Petzold (1981). "An efficient numerical method for highly oscillatory ordinary differential equations". In: *SINUM* 18.3, pp. 455–479.
- L. N. Trefethen and D. Bau III (1997). *Numerical linear algebra*. Vol. 50. SIAM.

WKB expansion /1

- Alternatively, build nonoscillatory (approx) solution: WKB/Riccati defect correction
  - Wentzel–Kramers–Brillouin (WKB) expansion:
    Extract a small parameter $1/\omega_0$: let $\omega(t) = \omega_0 \Omega(t)$, $\omega_0 \gg 1$, $\Omega(t)$ unit size,

$$u''(t) + \omega_0^2 \Omega(t)^2 u(t) = 0$$

for both real and imag ω, u has exp behavior, so transform as $z(t) = e^{\omega_0 z(t)}$, $z'(t) = x(t)$,

$$x' + \omega_0 x^2 + \omega_0 \Omega^2 = 0,$$

then expand as power series in small param,

$$x_j(t) = \sum_{l=0}^{j} \omega_0^{-l} s_l(t)$$

match powers of ω_0, then "reabsorb": set $\omega_0 = 1$. Get

$$s_0 = \pm i\omega, \quad s_{l+1} = -\frac{1}{2s_0}\left(s_l' + \sum_{k=1}^{l} s_k s_{l+1-k}\right),$$

WKB expansion /2

- Usually applied analytically, used in quantum mechanics
- Recursion relation involves all previous terms $\rightarrow$ hard to analyze
- Asymptotic
- First few iterations of series (start from $+i\omega$):

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\begin{aligned}
x_0 &= i\omega, \\
x_1 &= i\omega - \frac{\omega'}{2\omega}, \\
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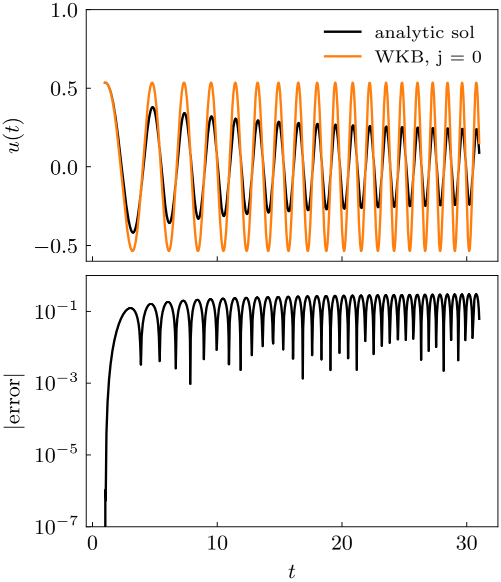

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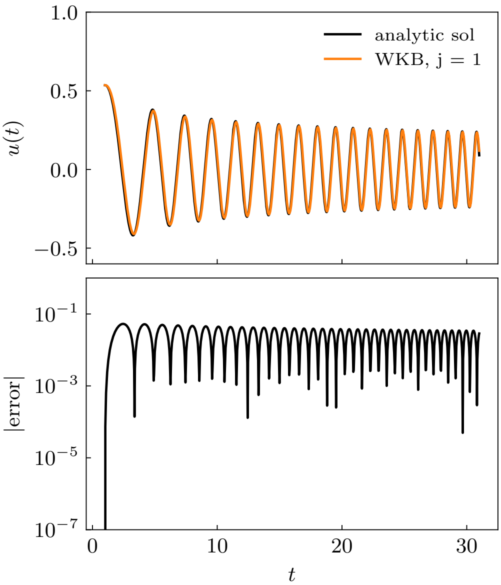

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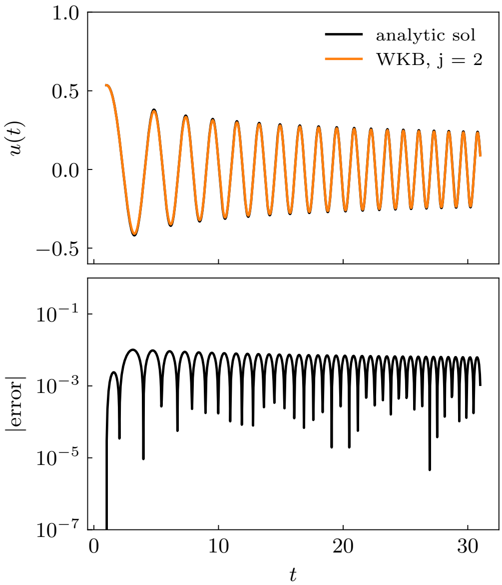

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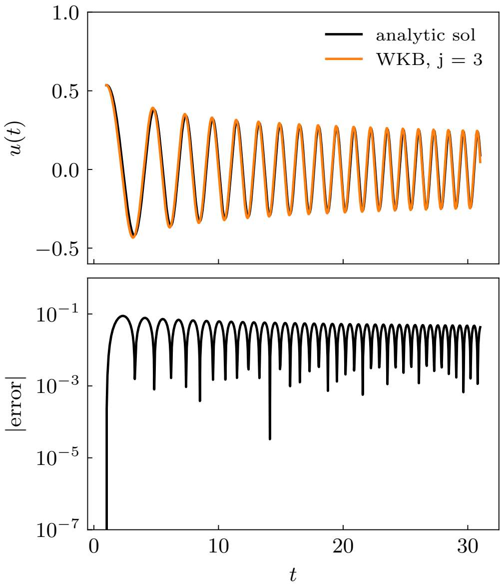


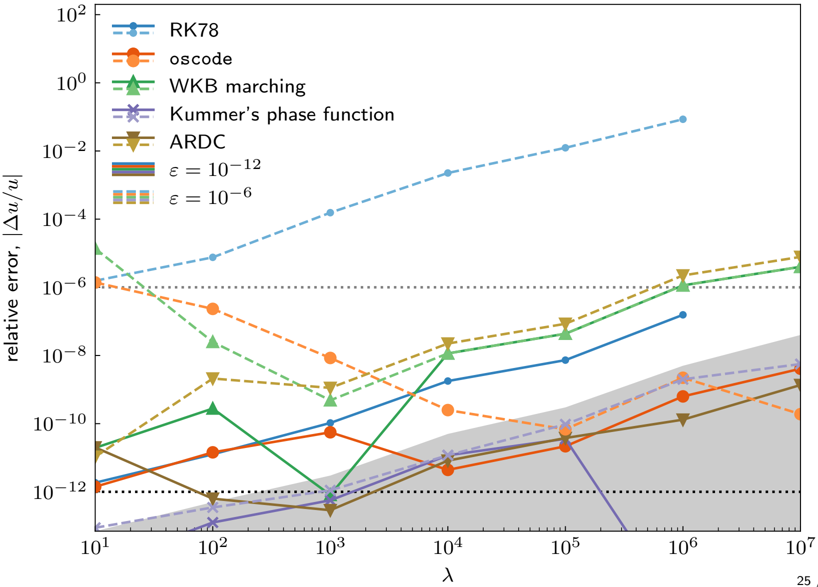
Some state-of-the-art oscillatory solvers

	ARDC/this work **Agocs** and Barnett (2022)	Kummer's phase function method Bremer, *ACHA* (2018)	`oscode` **Agocs**, Handley, et al., *Phys Rev Research* (2020)	WKB marching Körner et al., *JCAM* (2022)
high-order?	✓	✓	✗	✗
γ?	✓	✗[5]	✓	✗
code?	Python	Fortran 90	Python/C++	MATLAB
misc				need ω', ω'', $\ldots$, $\frac{\mathrm{d}^5\omega}{\mathrm{d}t^5}$

[5] Bremer, *ACHA* (2023) can be applied in this case, but no code $\rightarrow$ no comparison.

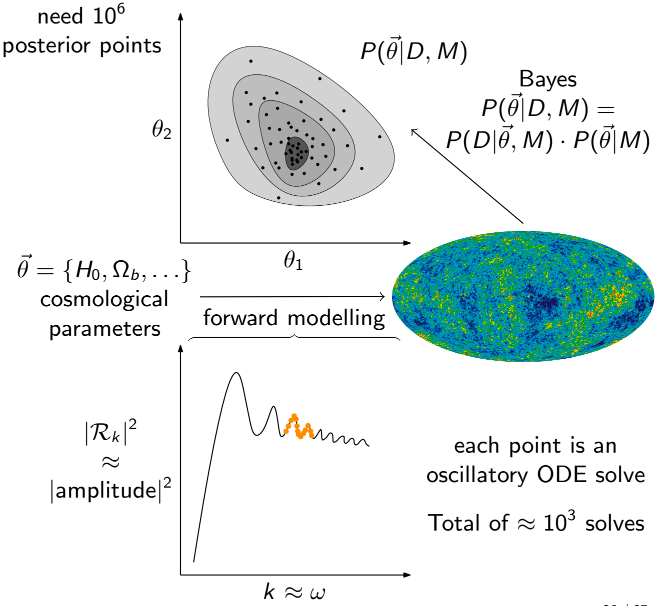
Comparison with standard & state-of-the-art solvers, convergence

- We used the Kummer's phase function method to compute a reference solution, therefore its reported accuracy (relative to spectral deferred correction), in **grey shading**, is an **upper limit** on the error



Motivation

- This ODE is extremely common in physics and math
  - inflationary cosmology
    - $\approx 10^9$ oscillatory ODE solves
  - 1D quantum mechanics
  - plasma physics, Hamiltonian dynamics, particle accelerators, electric circuits, acoustic and gravitational waves, ...
  - special function evaluation



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