Trapped acoustic waves and raindrops: High-order integral equation solution of the localized excitation of a periodic staircase

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With Alex Barnett, thanks to: Manas Rachh (FI), Leslie Greengard (FI), Eric Heller (Harvard)

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Scattering of a nonperiodic source from a periodic, corrugated surface

Questions, goals, and applications

- Interesting acoustic phenomena near corrugated surfaces, e.g. step-temples:
  - Sound travels “down” along stairs → trapped modes, propagating horizontally, evanescent perpendicular to stairs
  - Echo from footsteps sound like raindrops (Cruz et al, *Acta Acustica*, 2009)
- When do trapped modes exist? What is their dispersion relation?
- Compute single-frequency solution from single point excitation
- How does power in the system get distributed between trapped modes and outgoing radiation?
- Periodic surfaces have been exploited for their **waveguiding** properties:
  - Photonic crystals, acoustic metamaterials, diffraction gratings, antennae, anechoic chambers, amphitheaters, …
  - Fast, robust methods needed in **optimization** loops
  - →**Our method can have impact in the above applications**

El Castillo (“The Castle”),
a Mesoamerican step-pyramid in Chichen Itza, Mexico.

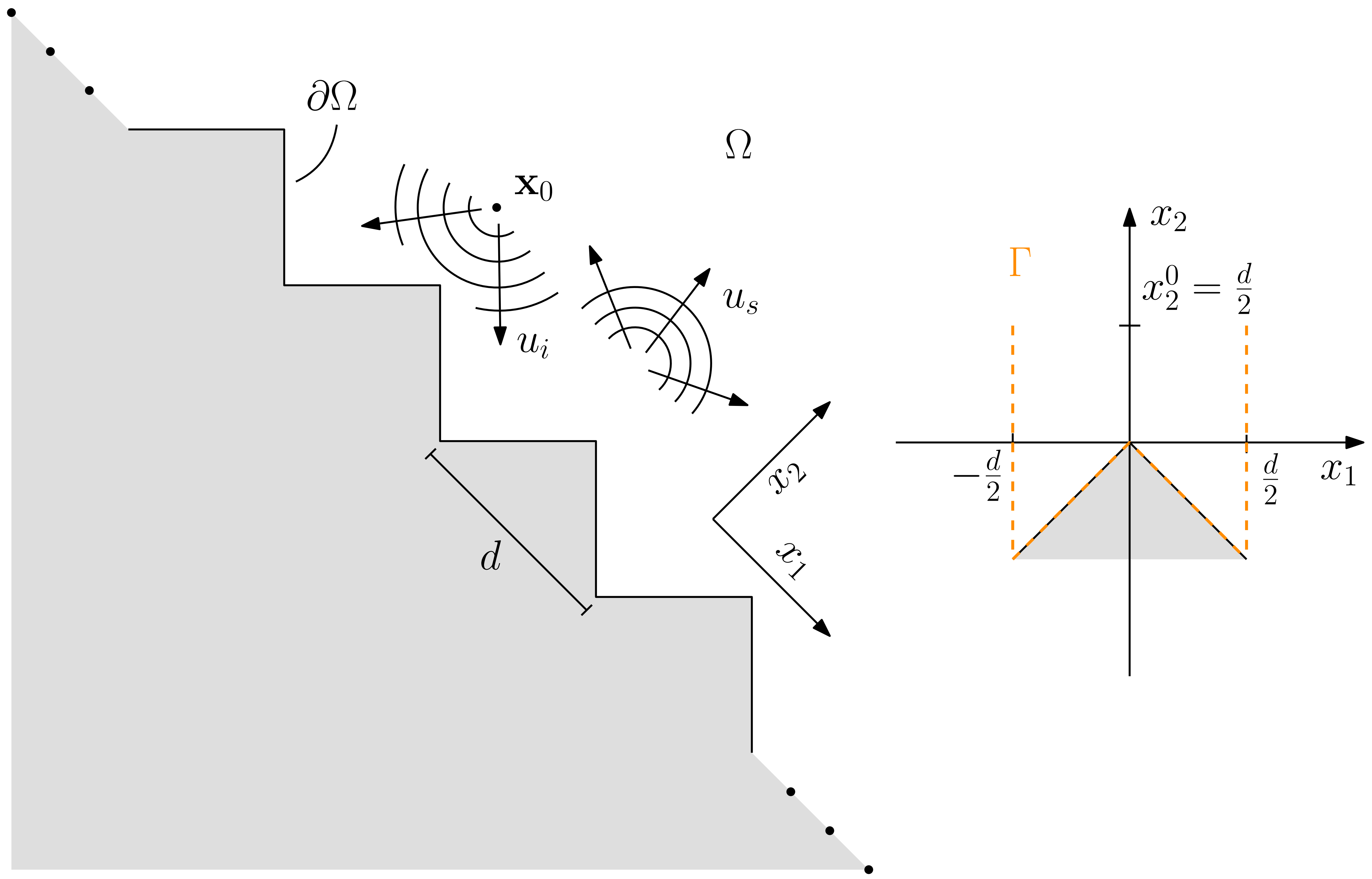
Why is this problem hard? Previous and new work

What's hard about this problem?

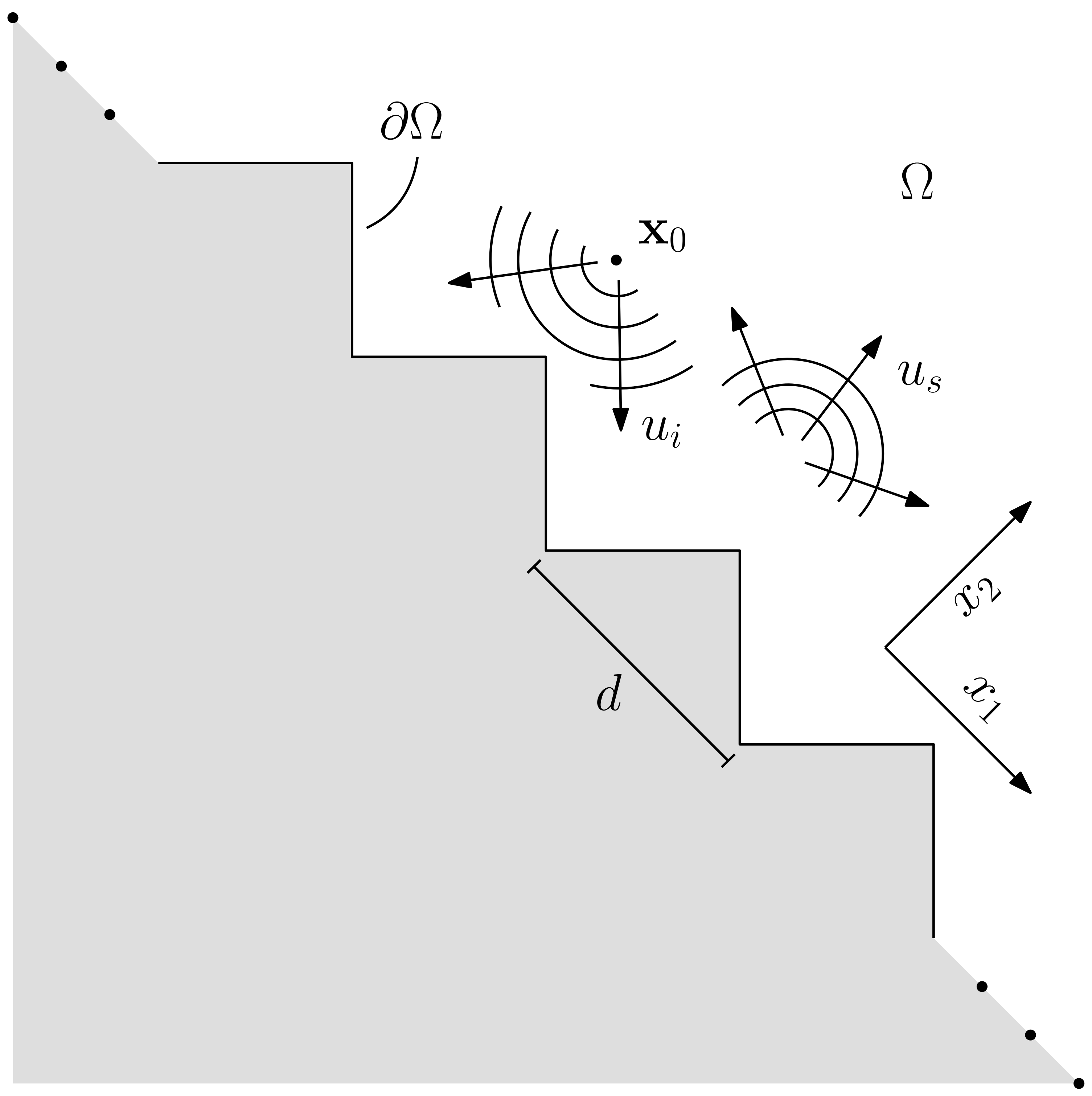
- Domain is infinite
- Periodic boundary → cannot truncate due to artificial reflections
- Nonperiodic source breaks periodicity → cannot reduce to single unit cell* (periodization)
- Corners introduce singularities

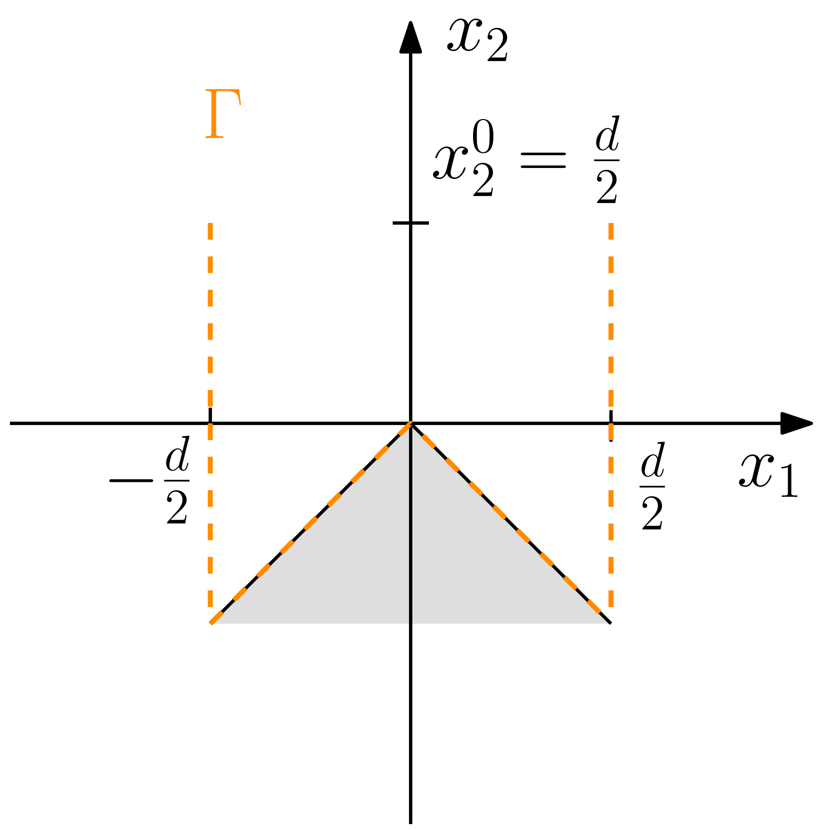
Previous work and what we are doing

- Finite differencing or finite elements methods
- Mesh-free methods: method of fundamental solutions, plane waves method
- Rayleigh methods based on the Rayleigh hypothesis
- Approximations, e.g. Helmholtz—Kirchhoff
- First **high-order accurate** scattering of a **nonperiodic source** from a **periodic surface with corners**: arXiv:2310.12486 (with Alex Barnett)
  - Boundary integral equation & method: $\mathcal{O}(N)$ instead of $\mathcal{O}(N^2)$, can deal with singularities and be accurate via high-order quadrature



Problem setup - quasiperiodic set of sources





$-(\Delta + \omega^2)u = \sum_{n=-\infty}^{\infty} e^{in\kappa d}\delta(\mathbf{x} - \mathbf{x}_0 - n\mathbf{d})$	in Ω,	PDE (Helmholtz)
$u_n = 0$	on $\partial\Omega$,	boundary condition (Neu)
$u(x_1 + nd, x_2) = \alpha^n u(x_1, x_2)$	$(x_1, x_2) \in \Omega$,	quasiperiodicity
$u(x_1, x_2) = \sum_{n\in\mathbb{Z}} c_n e^{i(\kappa_n x_1 + k_n x_2)}$,	$x_2 > x_2^0$	radiation condition

- $\mathbf{x} = (x_1, x_2)$ position vector, $\mathbf{d} = (d, 0)$ lattice vector.
- u_i is the incident, u_s is the scattered wave
- $u = u_i + u_s$ is the total solution
- κ is the **horizontal (on-surface) wavenumber**
- $u_n := \mathbf{n} \cdot \nabla u$ normal derivative in the outward sense
- If there are multiple sources, quasiperiodicity condition ensures the solution obeys the symmetry of the boundary
- The solution accrues an overall **(Bloch) phase** $\alpha = e^{i\kappa}$ over one **period** d.
- Set of possible horizontal wavevectors $\kappa_n = \kappa + 2\pi n/d$, $n \in \mathbb{Z}$, all lead to the same quasiperiodicity
- If the total wavevector is $\mathbf{k} = (\kappa_n, k_n)$, then $k_n = \sqrt{\omega^2 - \kappa_n^2}$ is the vertical wavevector (imaginary part always +ve)
  - Vertically propagating or evanescent
  - $k_n = 0$ are **Wood anomalies** (abrupt change in behavior)

Boundary integral formulation

- Use a single-layer potential (SLP) representation for the scattered wave:

$$u_s(\mathbf{x}) = \mathcal{S}\sigma = \int_\Gamma \Phi_{\mathrm{p}}(\mathbf{x},\mathbf{y})\sigma(\mathbf{y})\mathrm{d}s_{\mathbf{y}}, \quad \mathbf{x} \in \mathbb{R}^2,$$

  ensures u will satisfy the PDE.

- Using the appropriate **jump relations**, this gives the Fredholm integral equation

$$(I - 2D^{\mathrm{T}})\sigma = -2f \quad \text{on } \Gamma,$$

  where $f = -(u_i)_n|_\Gamma$ is the boundary data, and σ is the unknown density, and

$$D^{\mathrm{T}} = \int_\Gamma \mathbf{n}_{\mathbf{x}} \cdot \nabla \Phi_{\mathrm{p}}(\mathbf{x},\mathbf{y})\sigma(\mathbf{y})\mathrm{d}s_{\mathbf{y}} \quad \text{on } \Gamma\,.$$

- Solve by discretizing the integral eq with **Nystrom's method**:

  if $v_i^{(N)} = \{(u_n)_i\}_{i=1}^N$ are the values of u_n at a set of quadrature nodes $\{s_i\}_{i=1}^N$ on the boundary with weights $\{w_i\}_{i=1}^N$, then

$$v_i^{(N)} - \sum_{j=1}^N w_j \Phi_{\mathrm{p}}(s_i, s_j) v_j^{(N)} = f(s_i), \quad \forall i = 1,2,\ldots,N\,,$$

  v is the density σ evaluated on the boundary nodes.

- u can then be reconstructed anywhere using the SLP.

$$u_s(t) = \sum_{j=1}^N w_j \Phi_{\mathrm{p}}(t, s_j) v_j^{(N)}$$

D. Colton and R. Kress, *Inverse Acoustic and Electromagnetic Scattering Theory*

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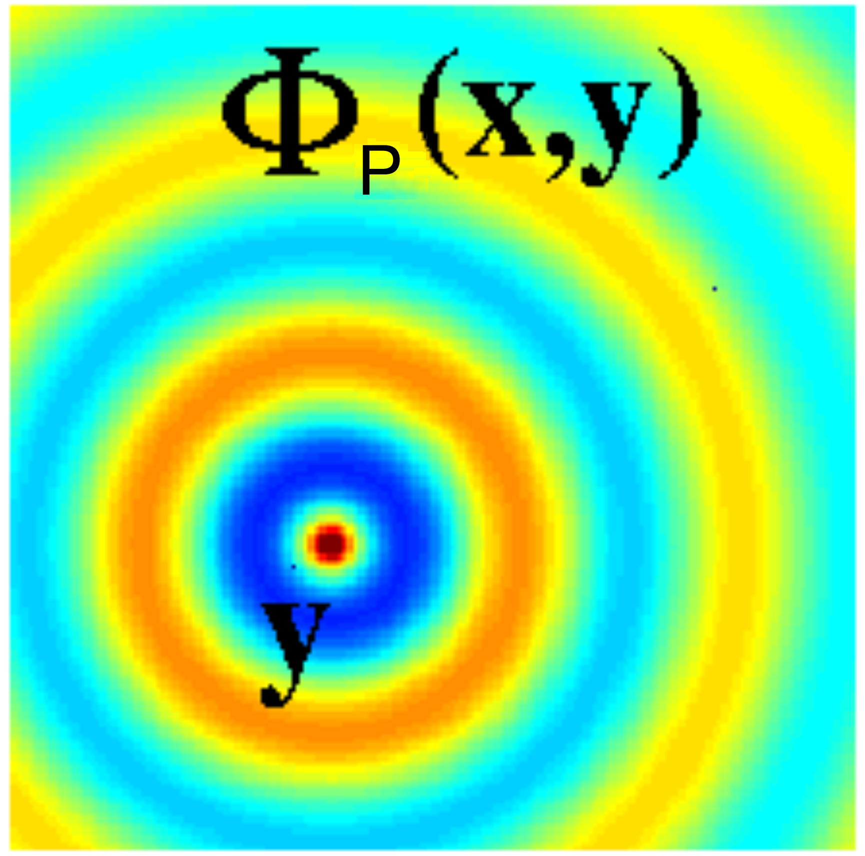

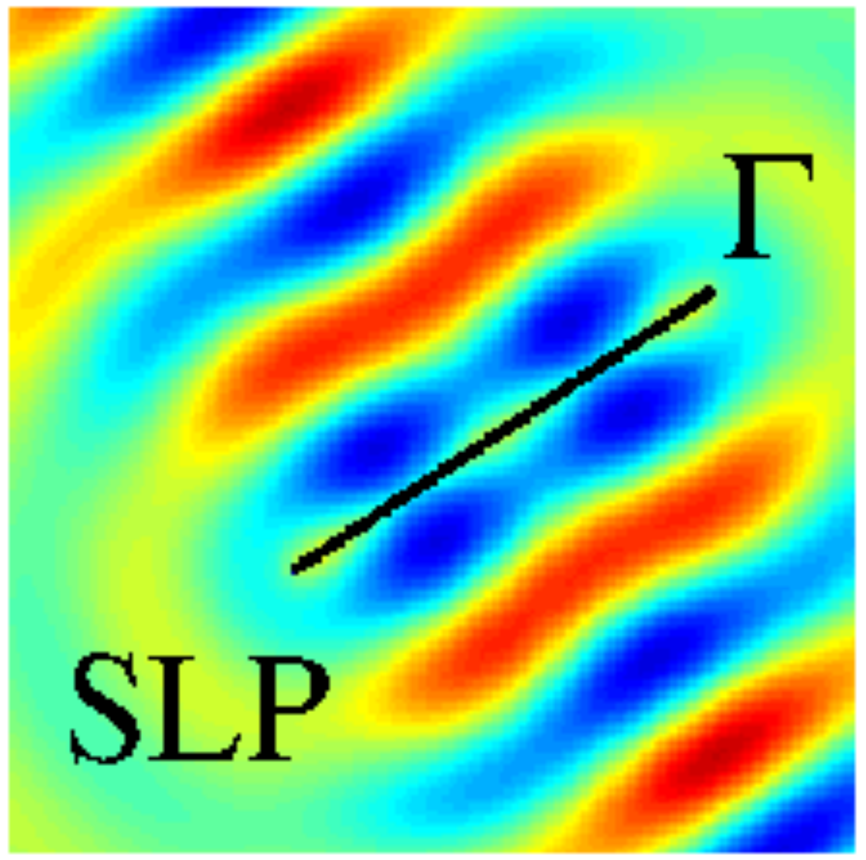

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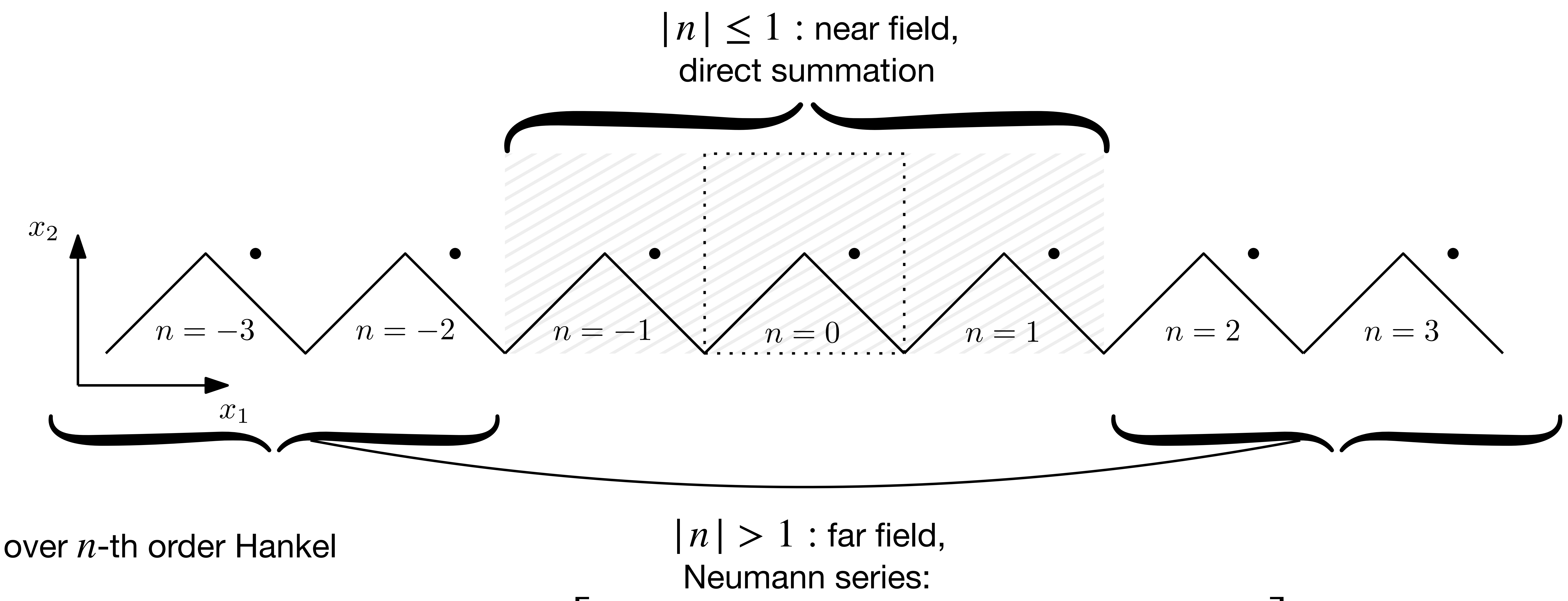
Periodization

- Reduce computation to the unit cell by using the **quasiperiodic Green's function**, $\Phi_{\mathrm{p}}(\mathbf{x},\mathbf{y})$, where $\mathbf{x}$ is the target's, $\mathbf{y}$ is the source's position vector:

$$-(\Delta+\omega^2)\Phi_{\mathrm{p}}(\mathbf{x},\mathbf{0}) = \sum_{n=-\infty}^{\infty} \alpha^n \delta(x_1 - nd)\delta(x_2)$$

$$\Phi_{\mathrm{p}}(\mathbf{x},\mathbf{0}) = \frac{i}{4}\sum_{n=-\infty}^{\infty} \alpha^n H_0^{(1)}\left(\omega\sqrt{(x_1-nd)^2+x_2^2}\right)$$

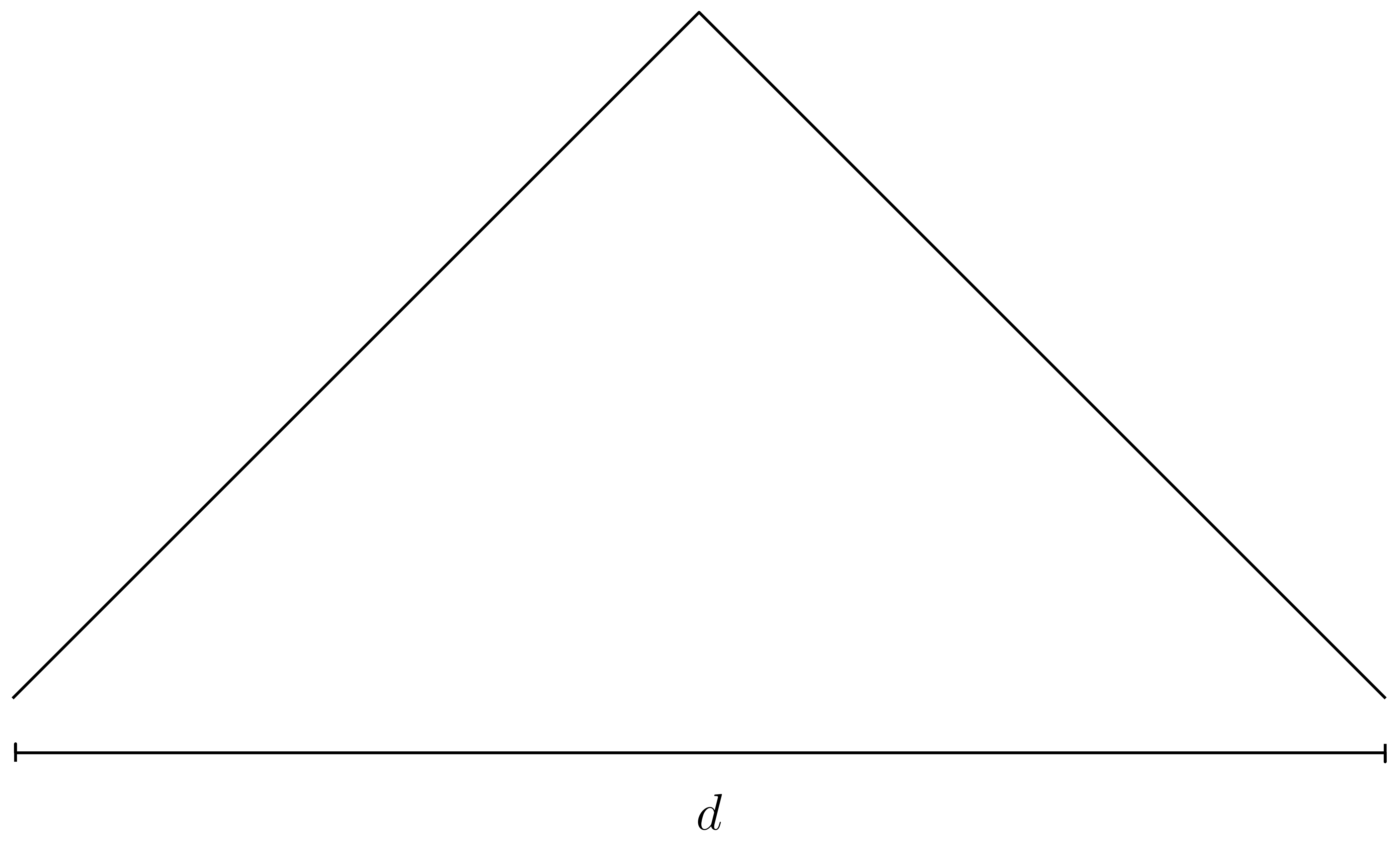


$$\Phi_{\mathrm{p,far}}(\mathbf{x},0) = \frac{i}{4}\left[S_0(\omega,\kappa)J_0(\omega,\mathbf{x}) + 2\sum_{n=1}^{\infty} S_n(\omega,\kappa)J_n(\omega,\mathbf{x})a(\mathbf{x})\right]$$

- The $S_n(\omega,\kappa)$ are **lattice sums** involving sums over n-th order Hankel functions
  - Computed once per ω,κ
  - Slowly convergent → use integral representation (Yasumoto and Yoshitomi, IEEETAP, 1999)
  - Only convergent in a disc → only use it inside unit cell

Boundary integral formulation; quadrature

- How to choose the quadrature nodes $\{s_i\}_{i=1}^N$?
- Integrand is singular at corners!
- $\rightarrow$ use panel quadrature with **adaptive corner refinement**:
    1. Lay down some equally sized initial panels
    2. Split corner-adjacent panels in a $1:(r-1)$ ratio ($r=2$, dyadic refinement shown)
    3. Lay down **Gauss—Legendre** quadrature nodes on panels.
- Quadrature coordinates **relative** to the nearest corner to avoid catastrophic cancellation
- No special rules (yet) for close evaluation



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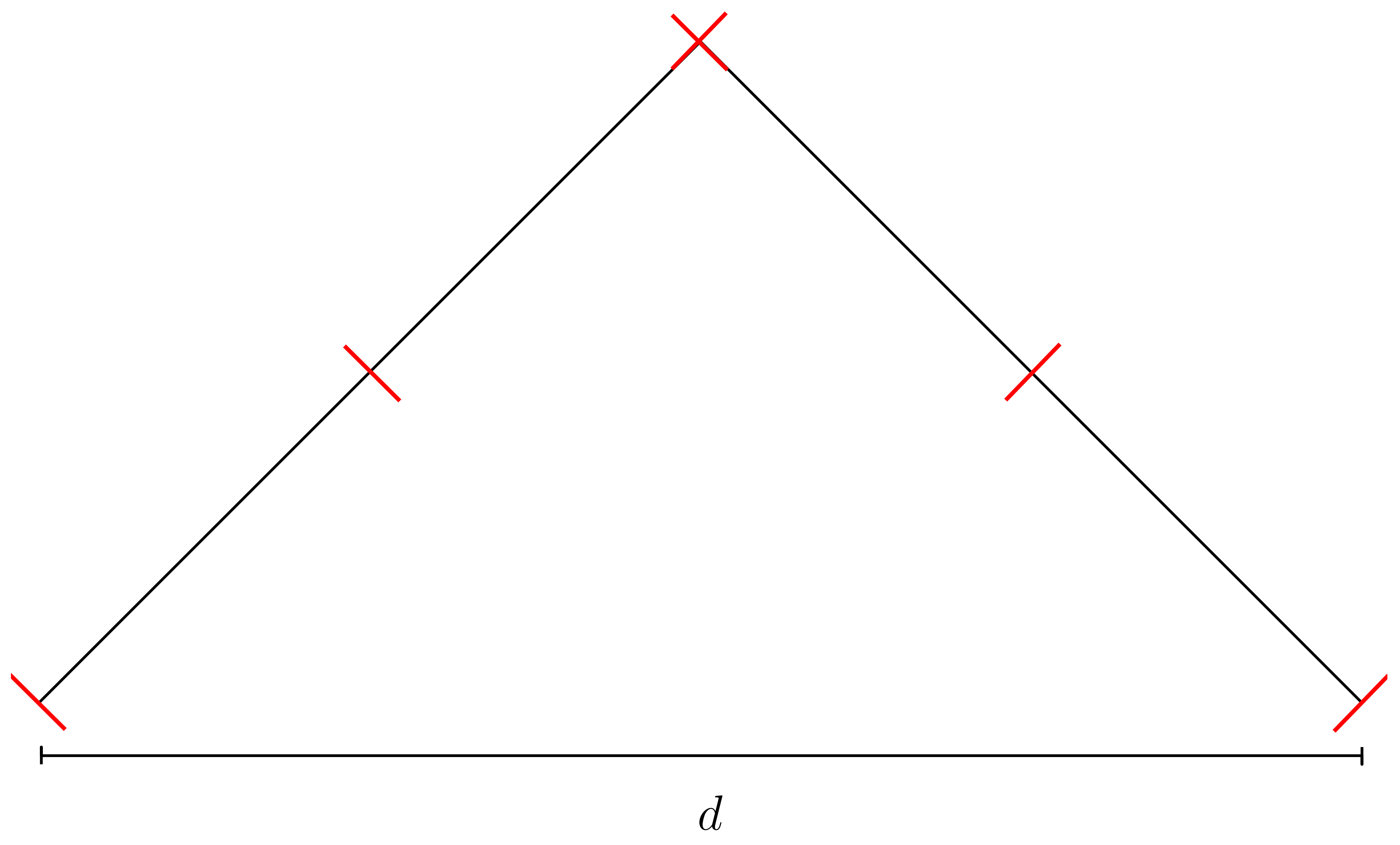

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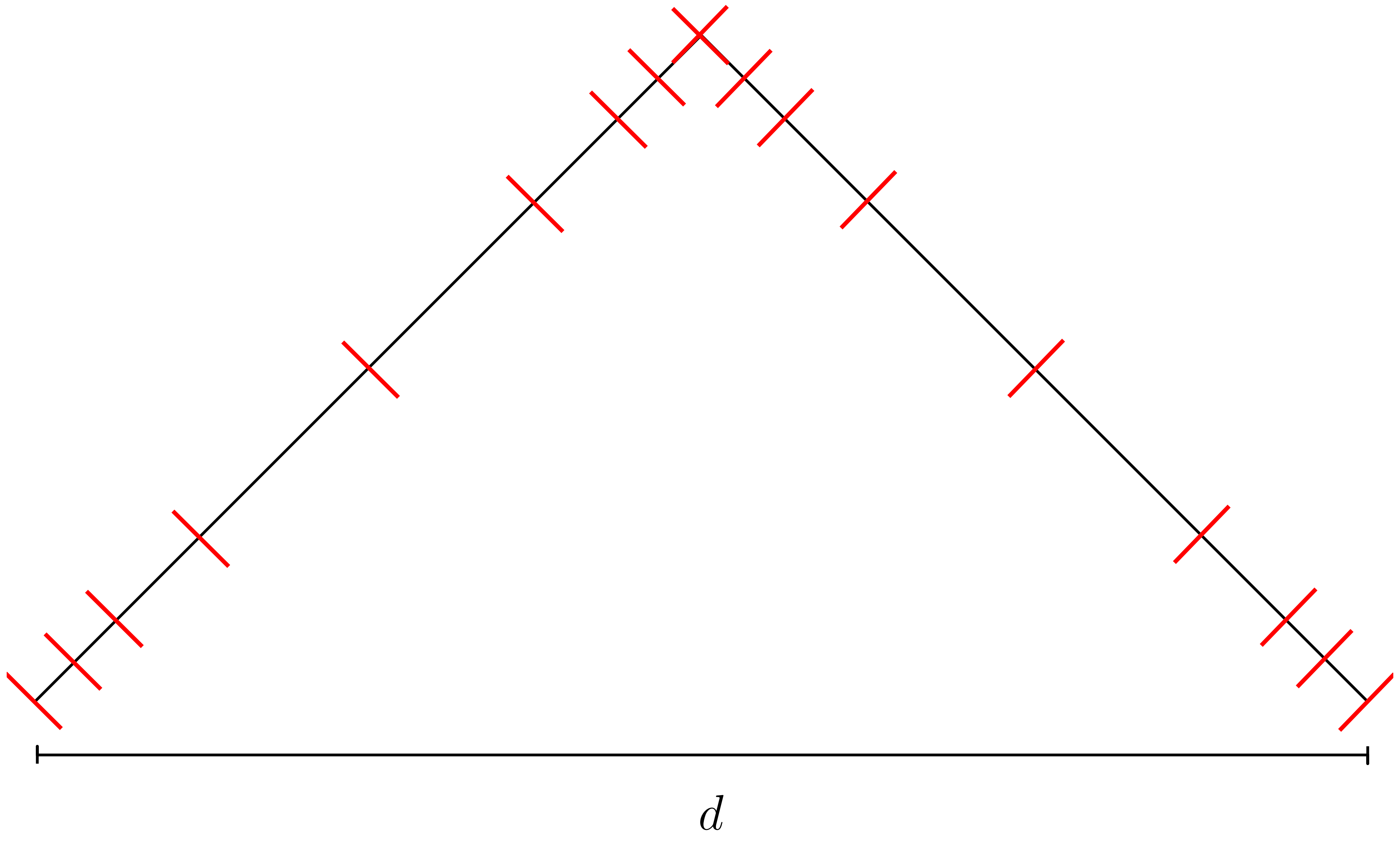

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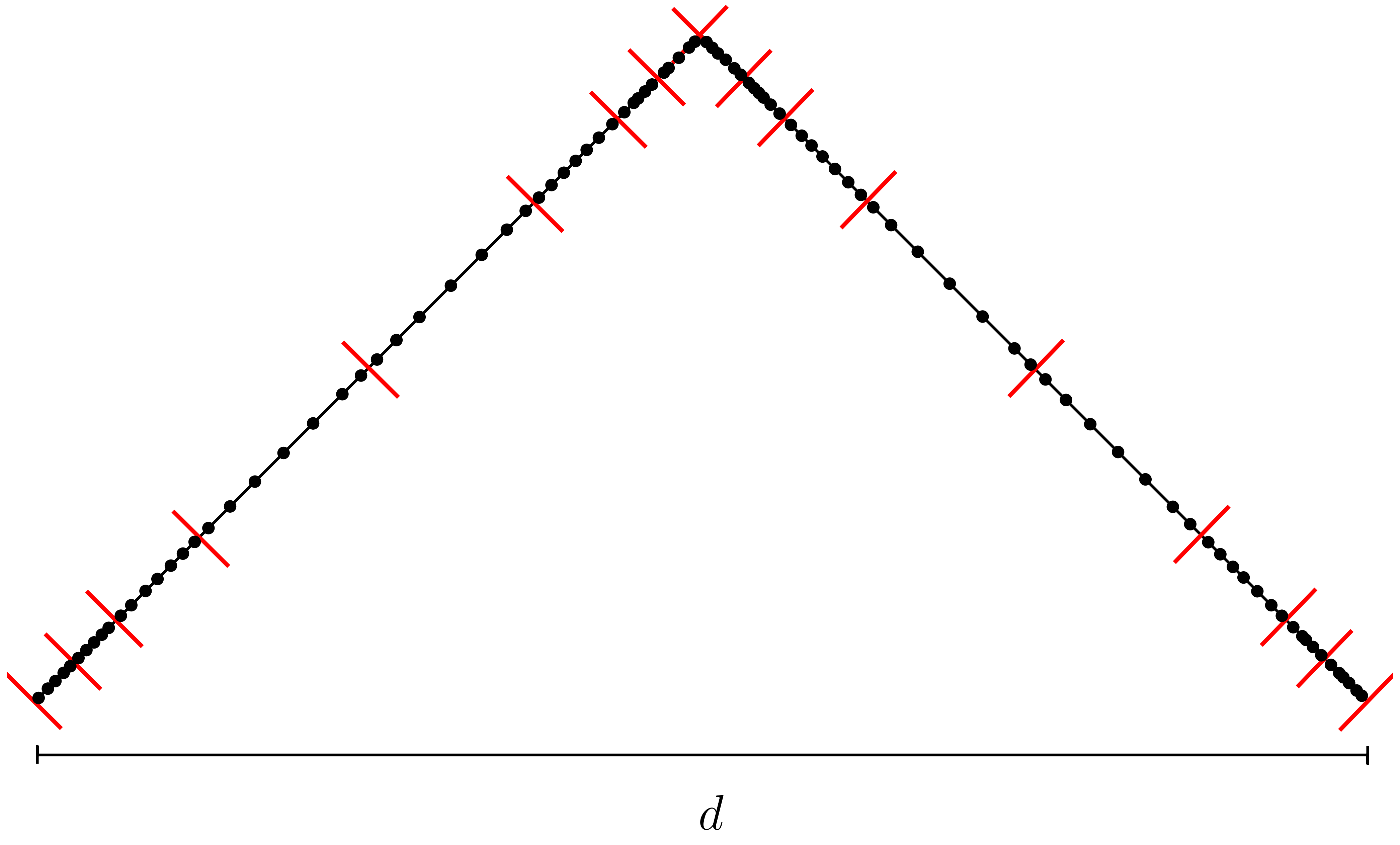


Reconstructing the solution

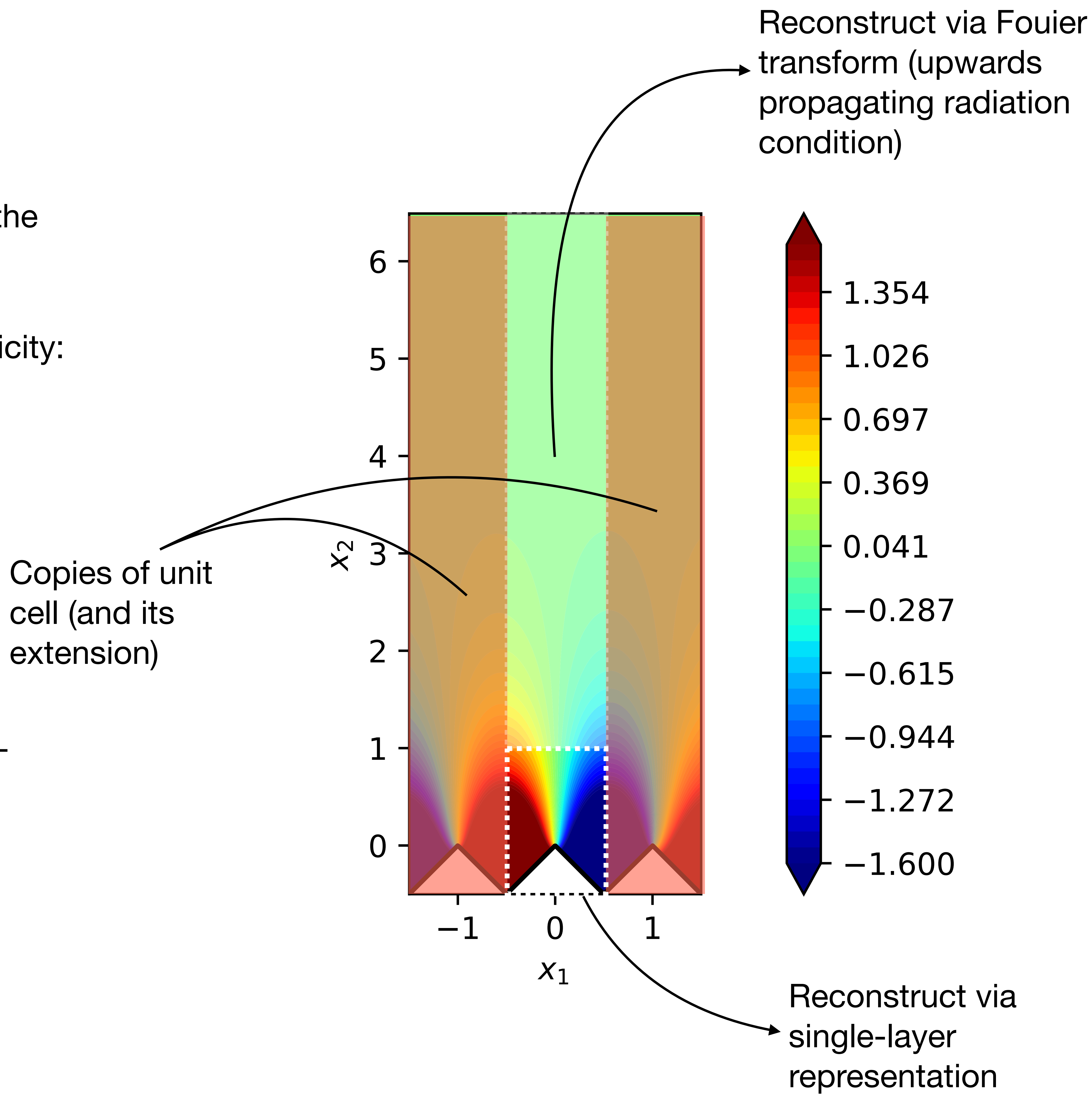
- Reconstructing u via the single-layer representation only works ~inside the unit cell, because lattice sum needed for $\Phi_{\mathrm{p}}(x, y)$ converges in a disc
  - Horizontally outside of unit cell (in neighboring cells), use quasiperiodicity:

$$u(x_1 + nd, x_2) = e^{in\kappa} u(x_1, x_2)$$

  - Vertically outside of unit cell (above), match solution to upwards propagating radiation condition via FFT:

$$u(x_1, x_2) = \sum_{n \in \mathbb{Z}} c_n e^{i\kappa_n x_1 + k_n x_2}, \quad x_2 > x_2^{(0)} = \frac{d}{2}$$

$$u(x_1, x_2) e^{-i\kappa x_1} = \sum_{n \in \mathbb{Z}} c_n e^{2in\pi x_1} e^{ik_n x_2} = \sum_{n \in \mathbb{Z}} \tilde{c}_n e^{2in\pi x_1} \quad \rightarrow \text{DFT}$$



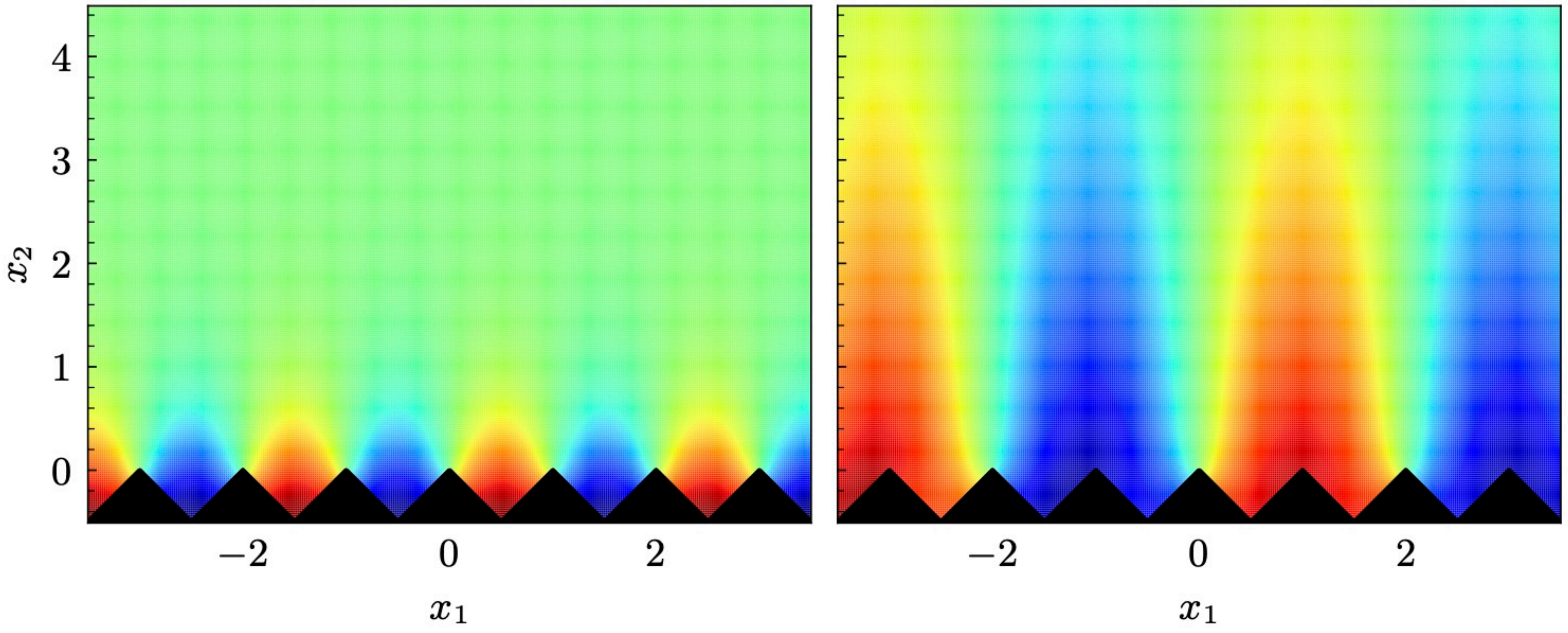
Finding trapped modes, chirp reconstruction via ray model

- Trapped modes occur when the Fredholm determinant is singular, i.e.

$$(I - 2D^{\mathrm{T}})\sigma = 0$$

has a nontrivial solution.

- Not a spurious resonance; this is a physical mode!
- D depends on κ, ω, so trapped modes only occur at some (κ, ω) combinations
- To find them: fix ω, sweep over all possible κ, $\kappa \in [-\pi, \pi]$ and do root finding (e.g. Newton's method)
- Compute:
  - Dispersion relation, $\omega(\kappa)$, of trapped modes
  - The group velocity of a trapped mode, $\frac{\mathrm{d}\omega}{\mathrm{d}\kappa}$, velocity at which the envelope of a wavepacket travels
  - **Ray model:** arrival time of different frequencies at El Castillo
    - Neglect: spreading along stairs in 3rd dimension; changes in amplitude; assume all trapped modes are excited



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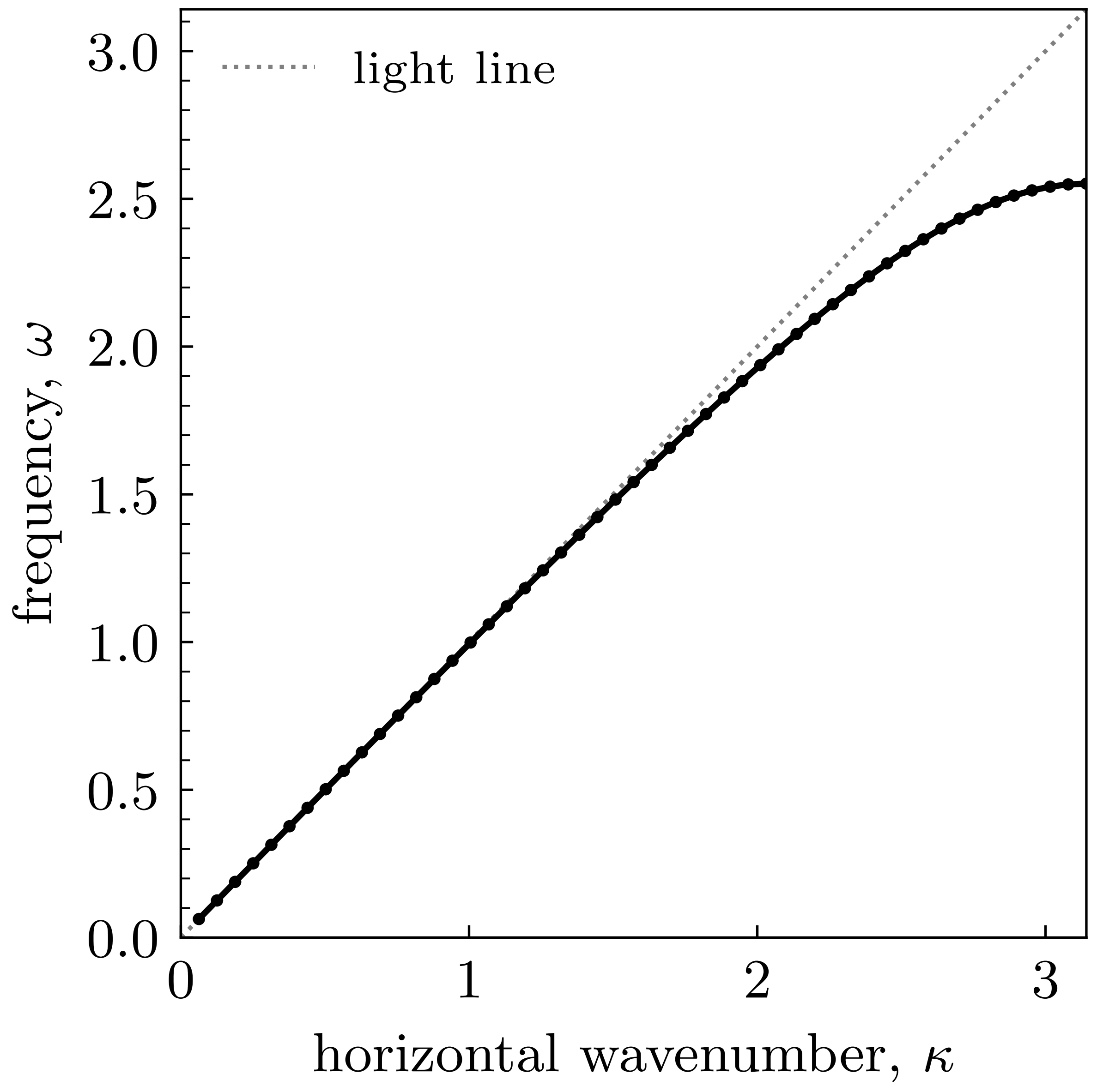

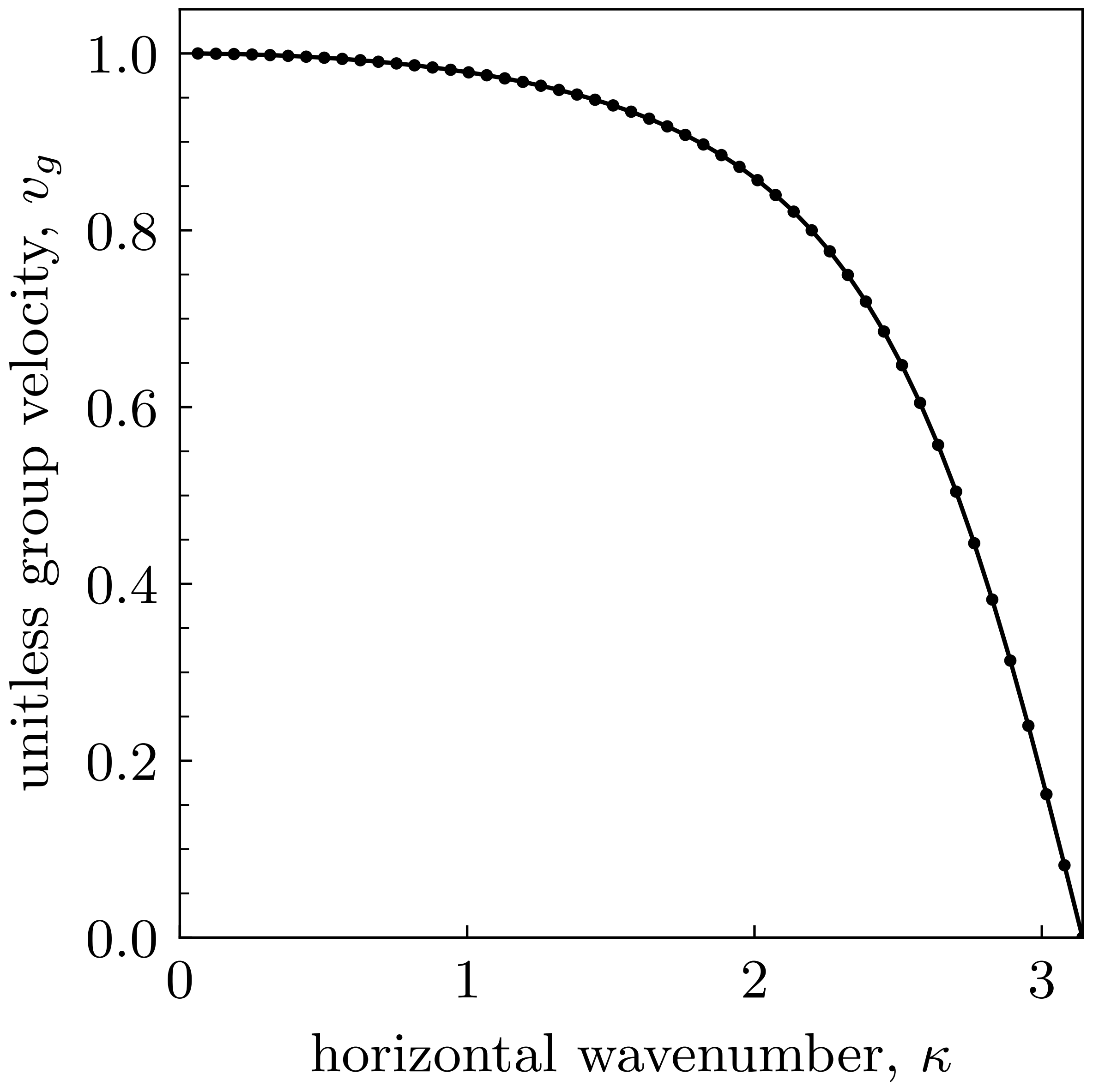

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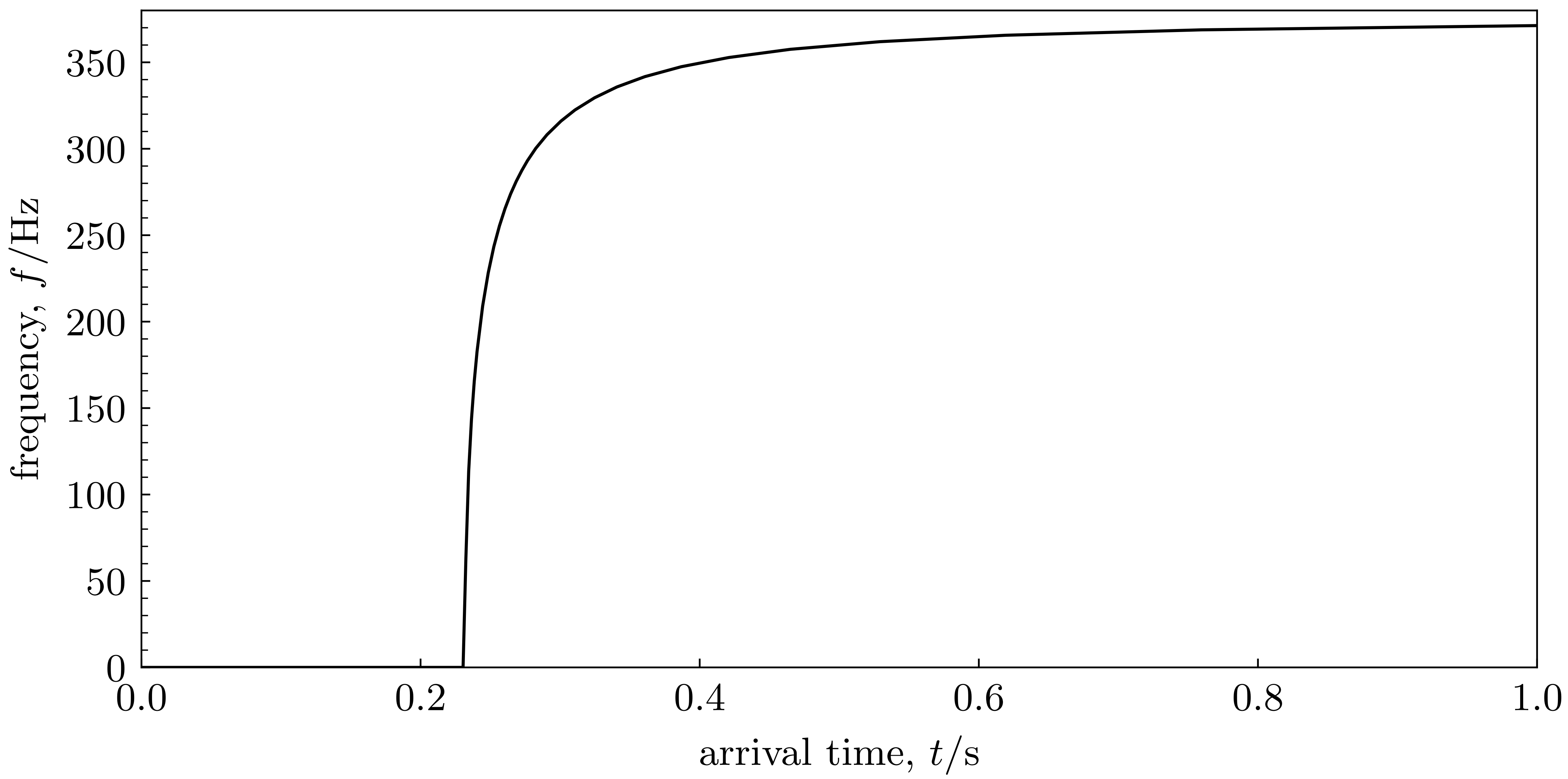


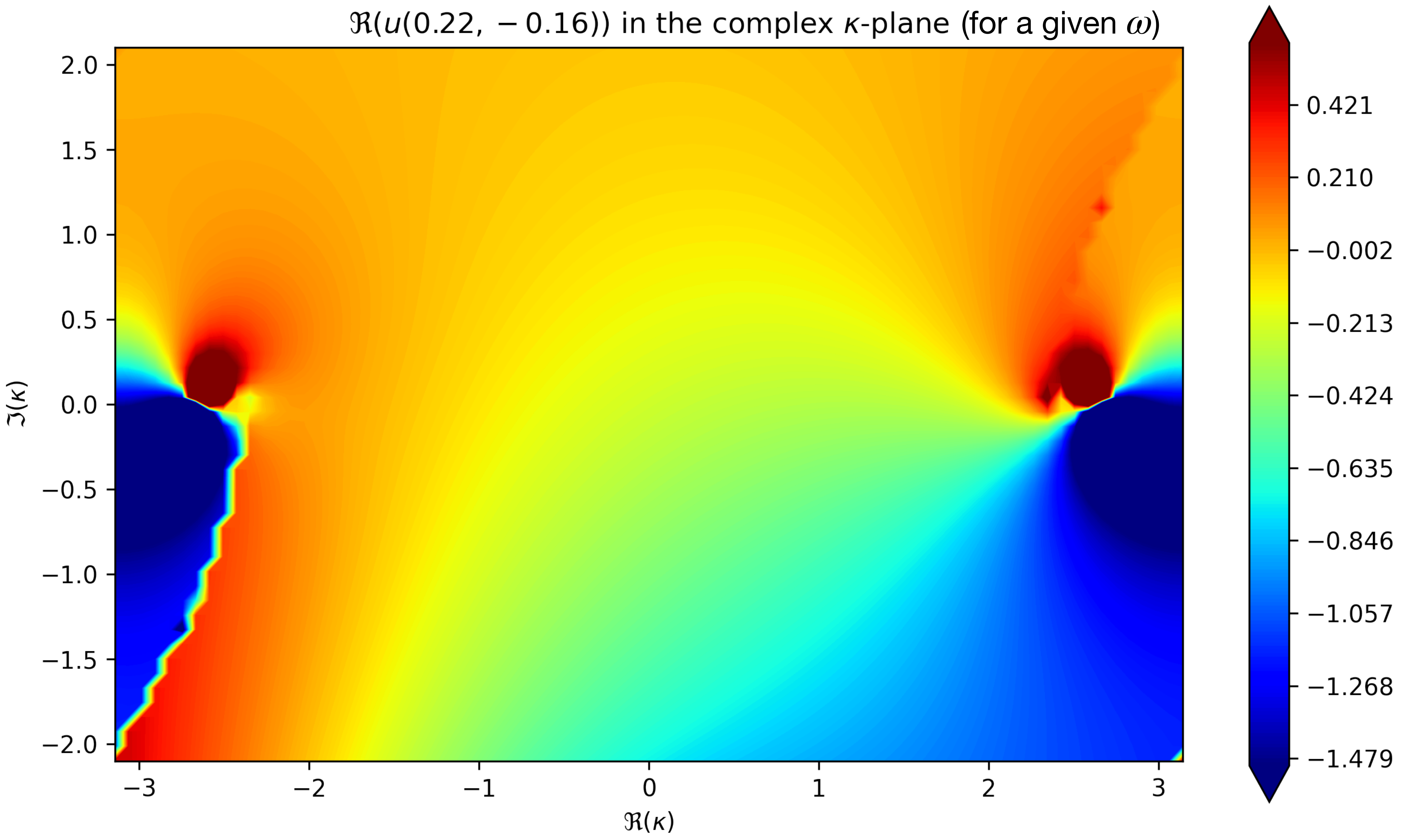
Array scanning / Floquet—Bloch transform

- **A neat trick**: write point source as an integral of quasiperiodic sets of point sources over the horizontal wavenumber κ

$$\delta(\mathbf{x}-\mathbf{x}_0) = \frac{d}{2\pi}\int_{-\pi/d}^{\pi/d}\sum_{n=-\infty}^{\infty} e^{in\kappa d}\delta(\mathbf{x}-\mathbf{x}_0-n\mathbf{d})\mathrm{d}\kappa,$$

  $\rightarrow$ the scattered wave from a single point source can be obtained by integrating $u_s(x,\kappa)$ in the first Brillouin zone, $\kappa \in [-\pi,\pi]$. (Munk and Burrell, IEEETAP, 1979)

- But, on real axis:
    - Branch cuts at Wood anomalies κ_{W}, squareroot singularity
    - Poles at trapped modes κ_{tr}
- Contour deformation (example path shown), sinusoidal with amplitude A, trapezoidal rule with P_{asm} nodes
- Direction of branch cuts/contour obeys the **limiting absorption principle:** $u(\mathbf{x},t) = u(\mathbf{x})e^{-i\omega t}$ correspond to outgoing waves



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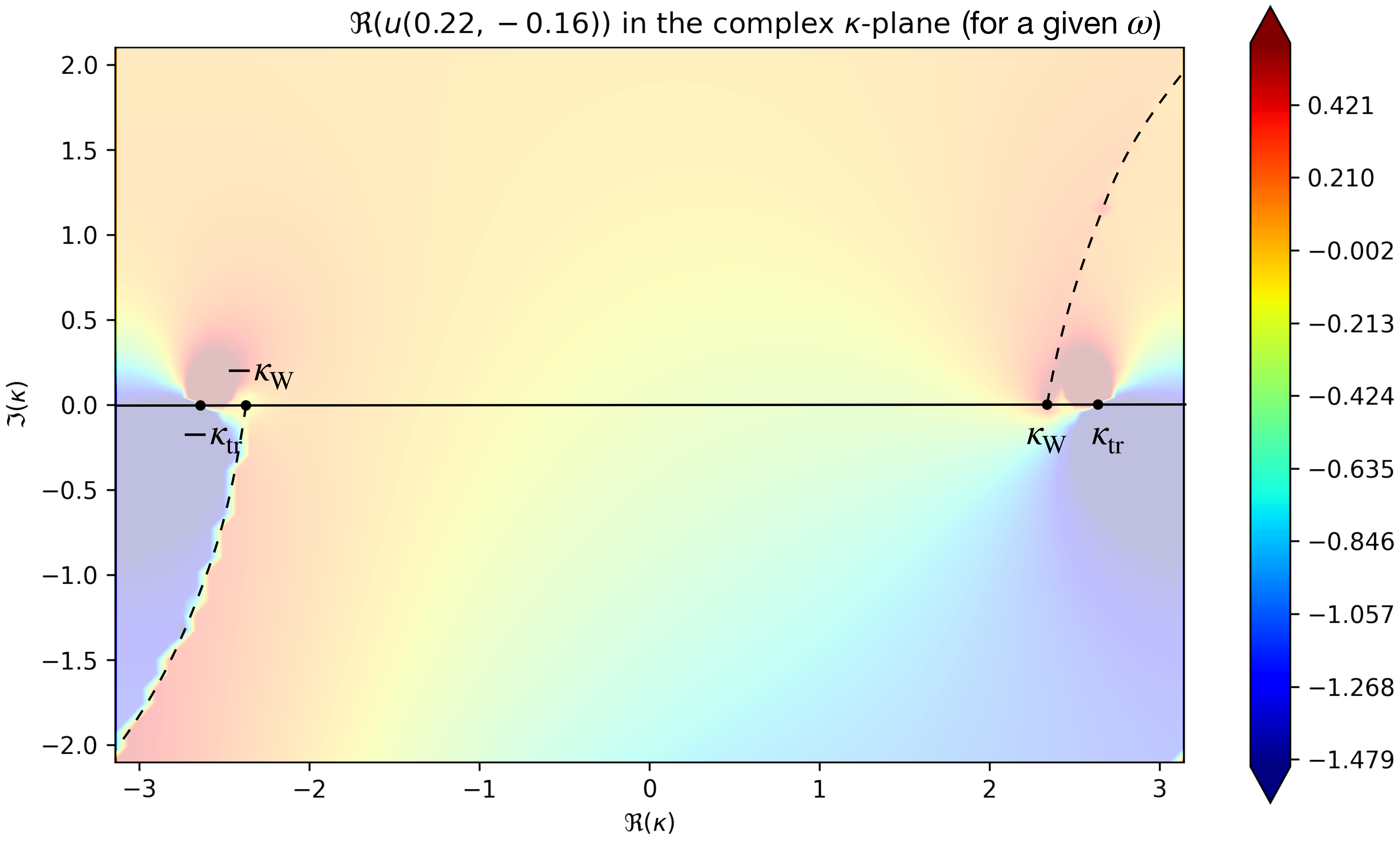

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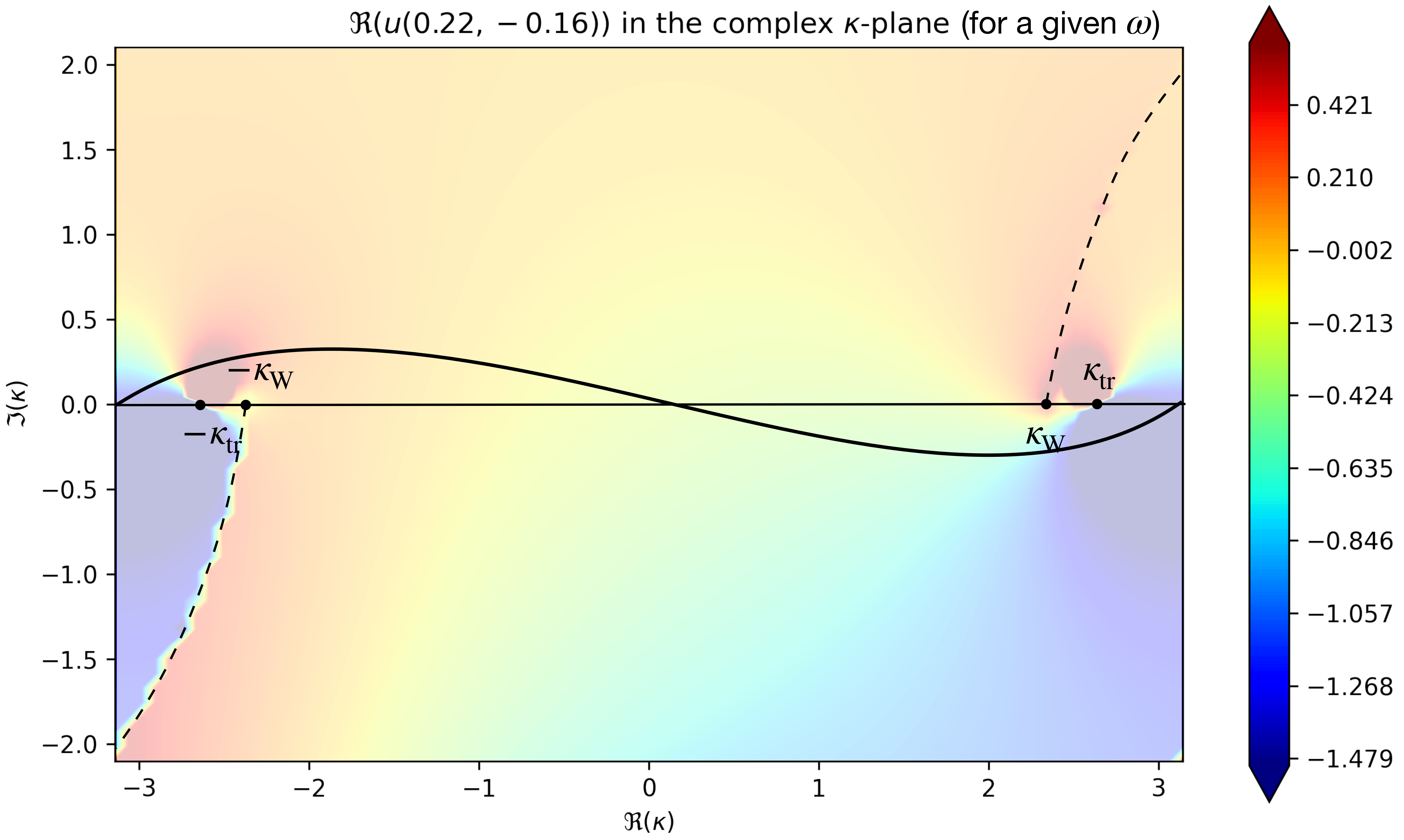

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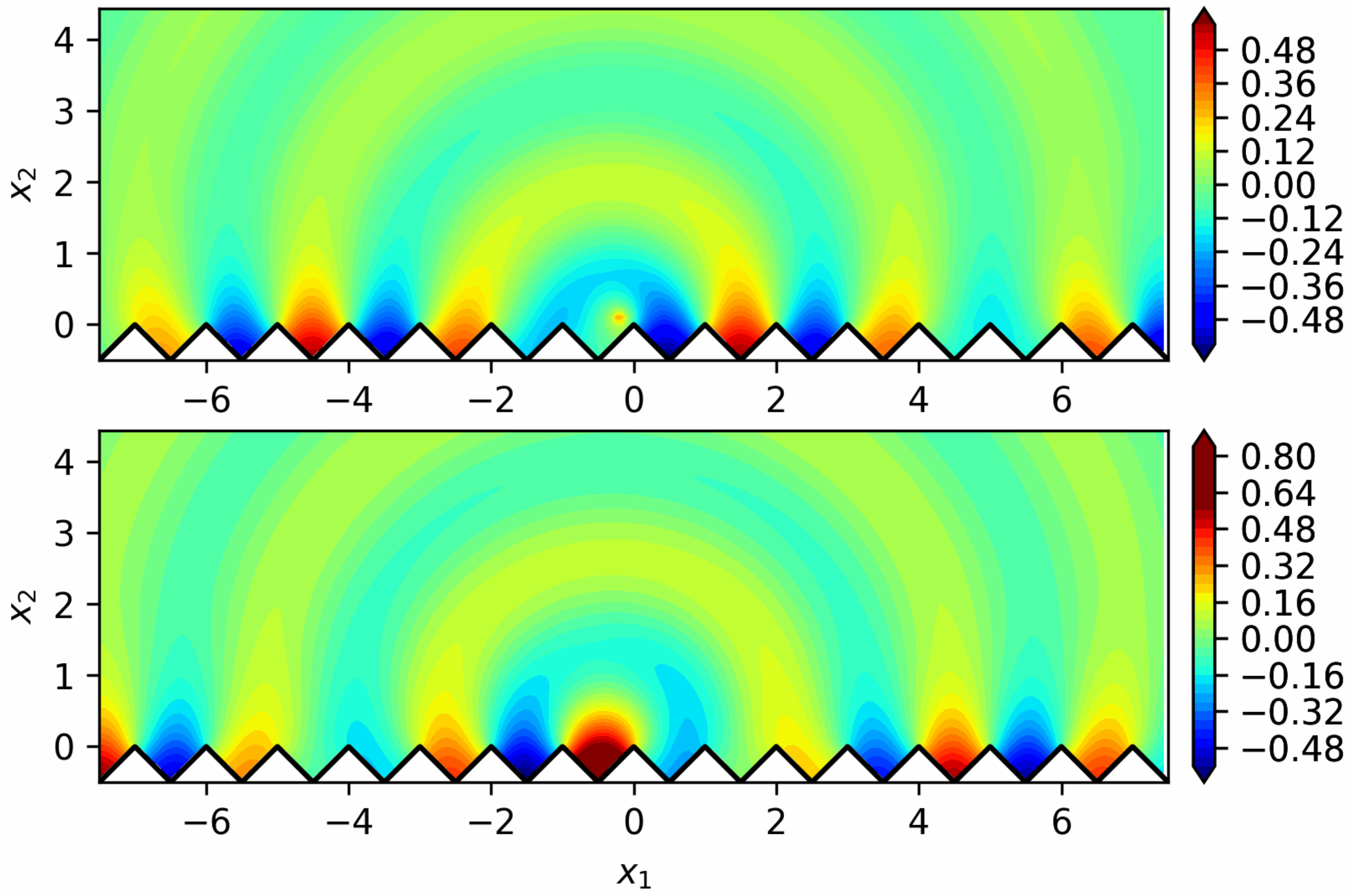
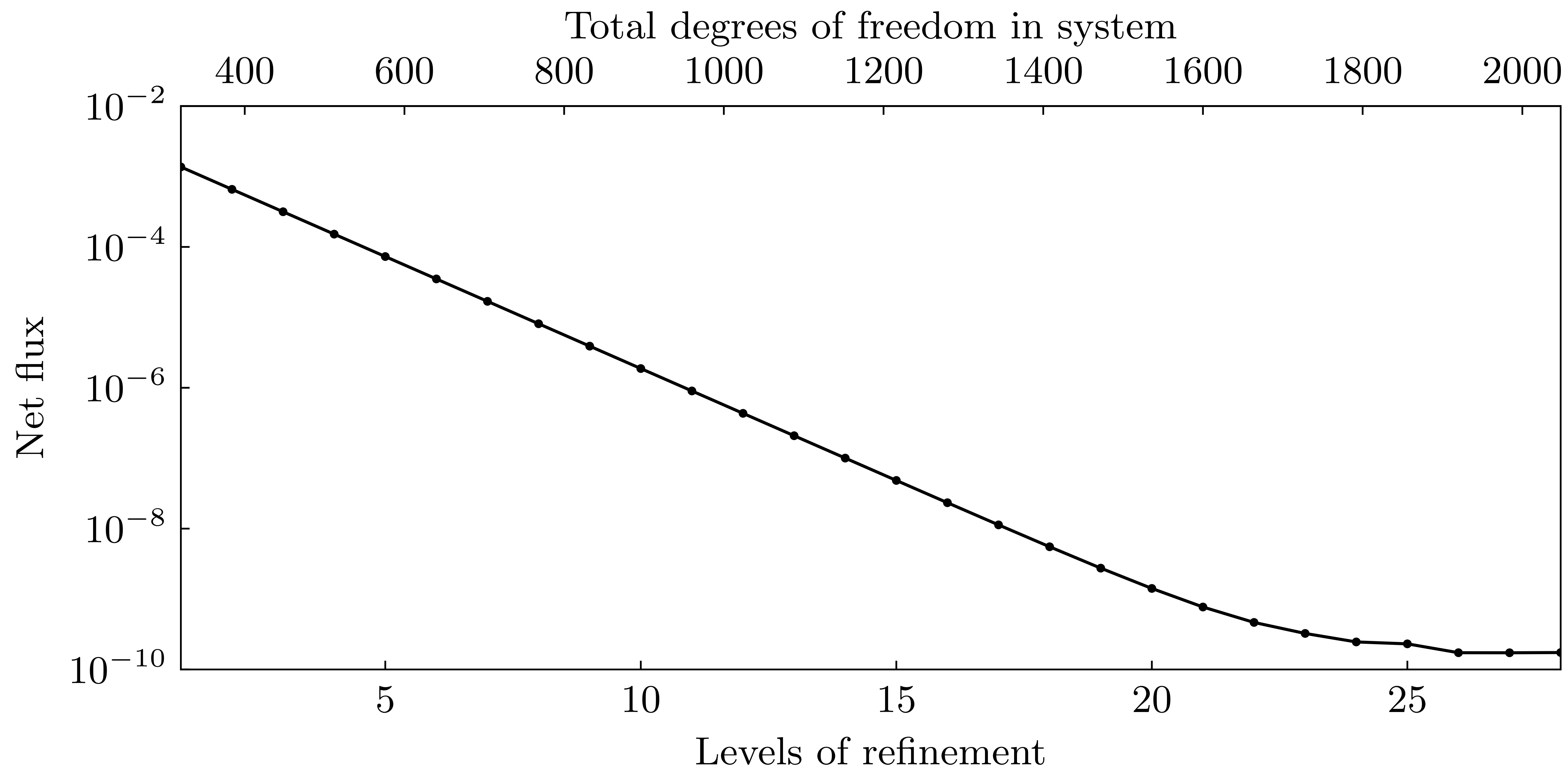
Time-propagation of the total field away from the source (for a single ω)



Convergence tests

- Analytic solution unknown and self-convergence can mislead → devise convergence test via conserved quantity
- **Net flux** (probability current in QM) **conserved** over a closed box: for an incoming plane wave, $\Im \int_\Gamma \bar{u} u_n \mathrm{d}s = 0$ (no source inside)
- How close is it to 0 numerically?
- Test convergence in the number of quadrature nodes along array scanning contour: how well can we reconstruct a single point source from a periodic array of point sources (i.e. $\Phi(\mathbf{x})$ from $\Phi_{\mathrm{p},\kappa}(\mathbf{x})$)?



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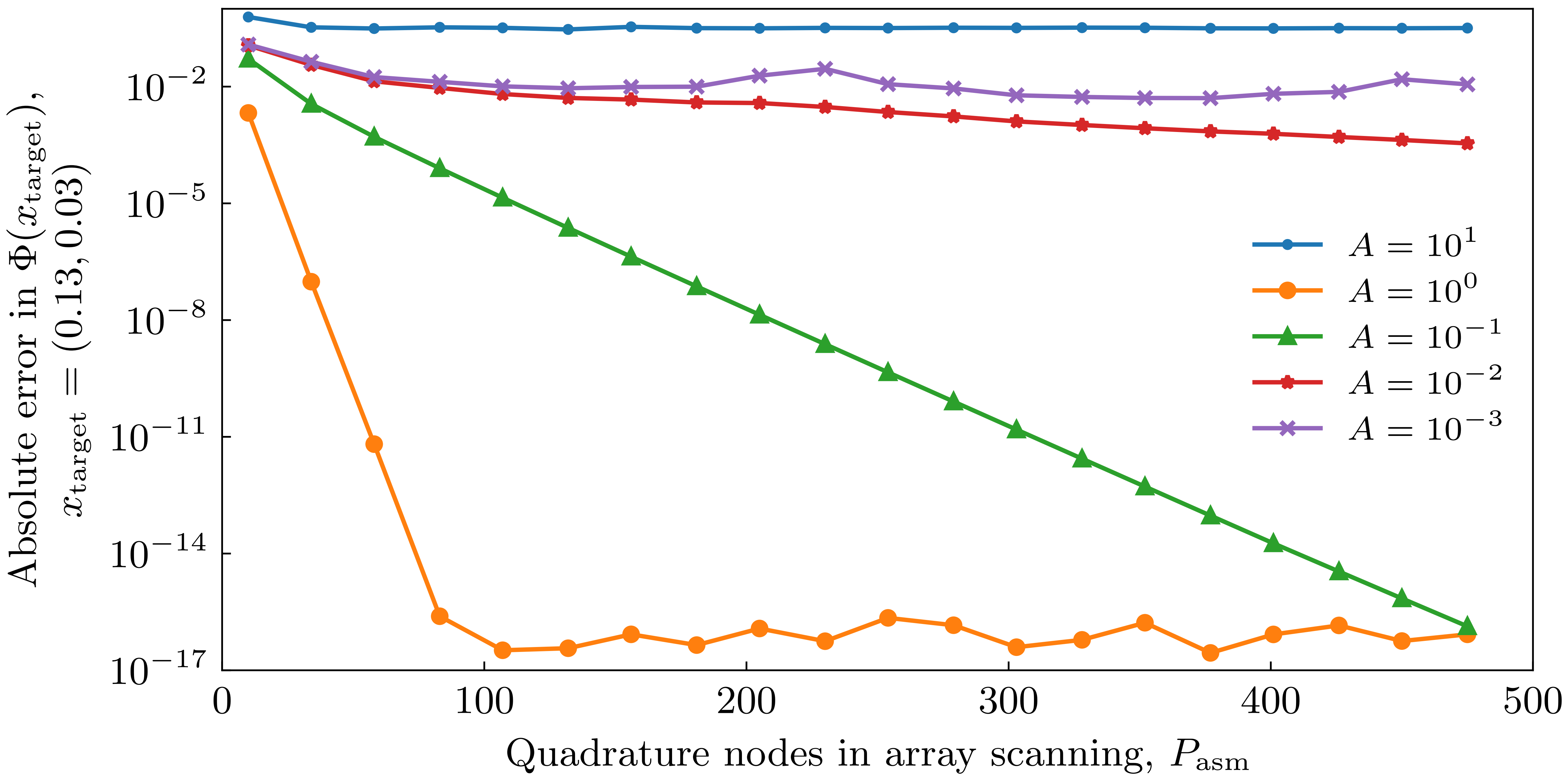


Power distribution in trapped modes

- What fraction of the total flux is transported in trapped modes?
- Claim: **in the far-away limit near the surface, only trapped mode remains,** i.e. only contribution to κ-integral will be from $\kappa = \kappa_{tr}$
- **Why?** Take solution in the limit of n (cell index) $\rightarrow \infty$,

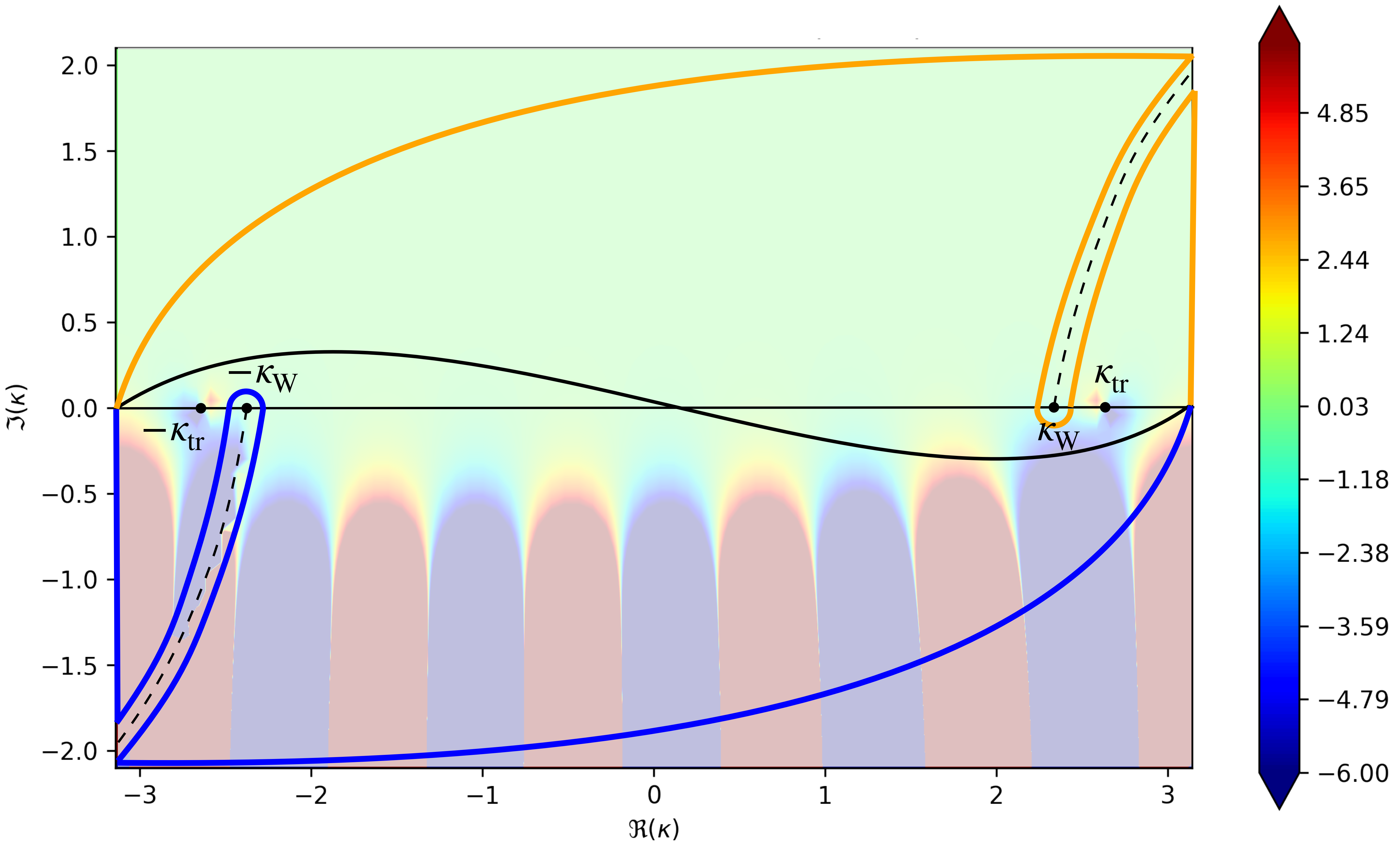
$$\lim_{n\to+\infty} u(x_1+nd, x_2) = \frac{1}{2\pi} \lim_{n\to+\infty} \int_{-\pi}^{\pi} u_\kappa(x_1,x_2) e^{in\kappa} \mathrm{d}\kappa.$$

  Close deformed contour in upper half plane $\rightarrow$ only residual of **right-hand pole** remains. Therefore,

$$\lim_{n\to+\infty} u(x_1+nd, x_2) = i\mathrm{Res}_{\kappa=\kappa_{tr}} u(x_1,x_2) \quad \text{up to a complex phase.}$$

  For $n \rightarrow -\infty$, **residue of left-hand pole dictates.**

- Compute residues numerically, on a small circle around κ_{tr} with trapezoidal rule.



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- **Why?** Take solution in the limit of n (cell index) $\to \infty$,

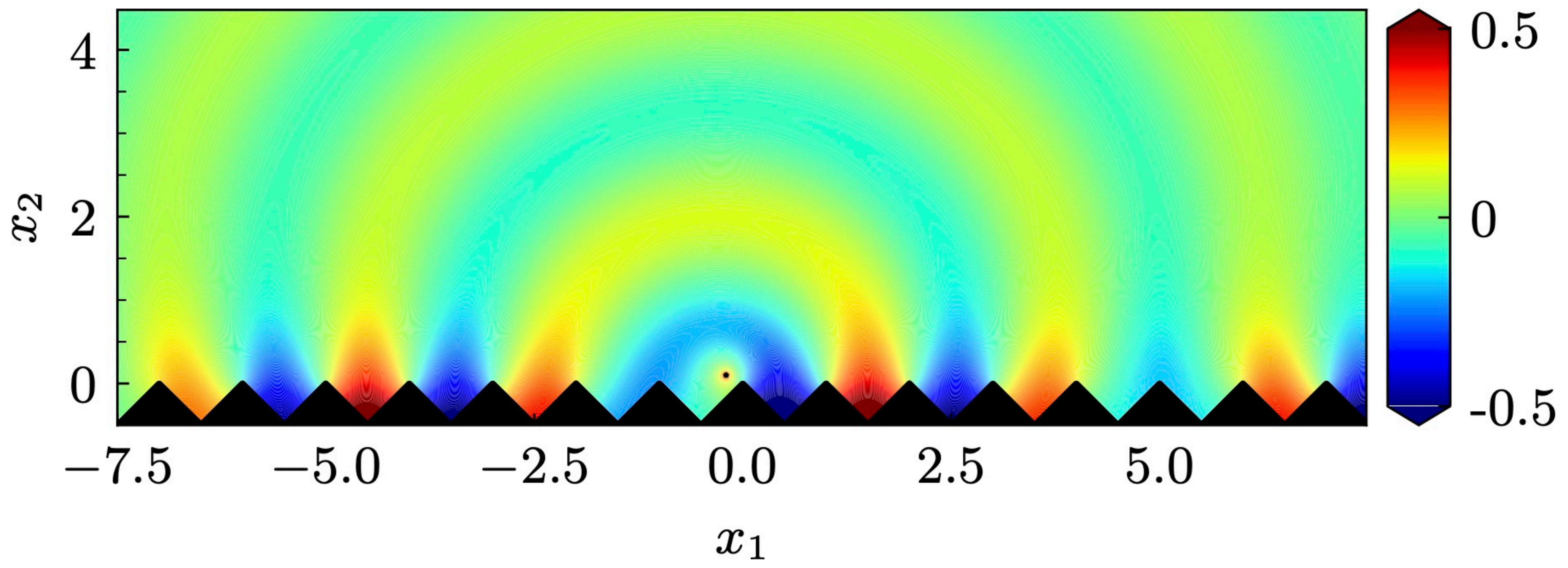
$$\lim_{n\to+\infty} u(x_1 + nd, x_2) = \frac{1}{2\pi} \lim_{n\to+\infty} \int_{-\pi}^{\pi} u_\kappa(x_1, x_2) e^{in\kappa} \mathrm{d}\kappa.$$

  Close deformed contour in upper half plane → only residual of **right-hand pole** remains. Therefore,

$$\lim_{n\to+\infty} u(x_1 + nd, x_2) = i\mathrm{Res}_{\kappa=\kappa_{\text{tr}}} u(x_1, x_2) \quad \text{up to a complex phase.}$$

  For $n \to -\infty$, **residue of left-hand pole dictates.**

- Compute residues numerically, on a small circle around κ_{tr} with trapezoidal rule.

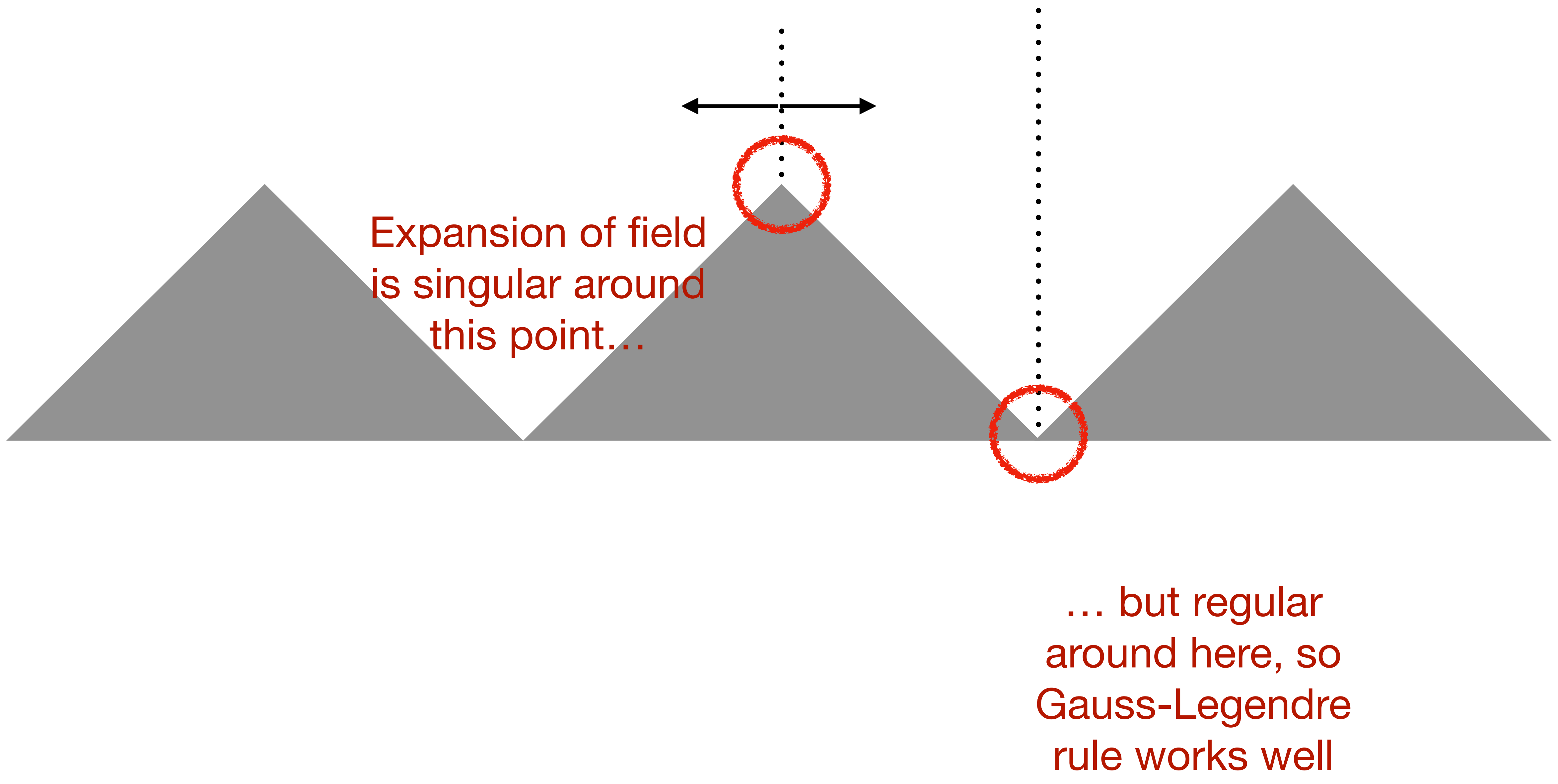


Power distribution in trapped modes

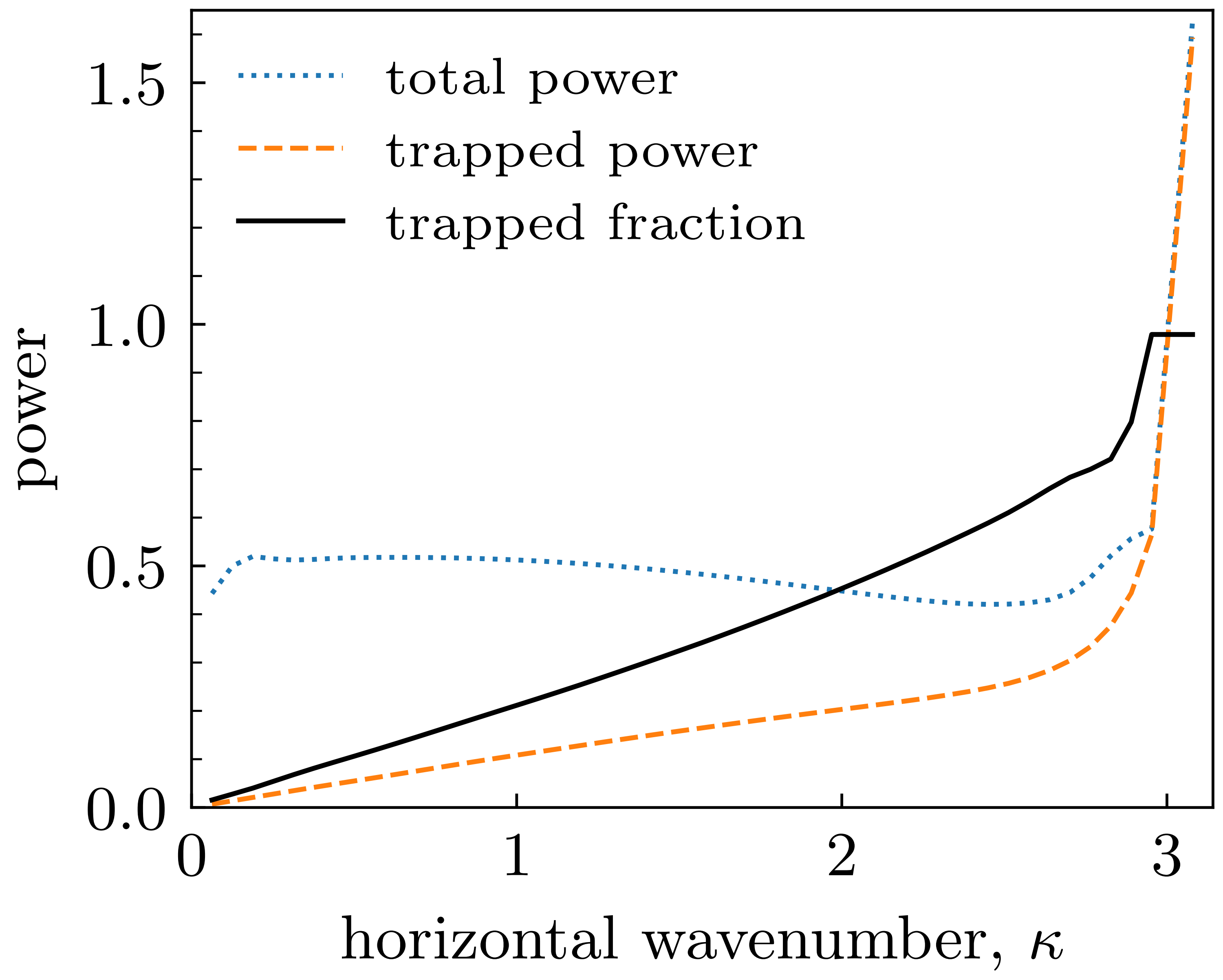
- We reconstruct the field $u(x)$ at an infinitely far unit cell on the right/left (up to a phase) by taking its residue around the trapping wavenumber $\pm\kappa_{\text{tr}}$
- Then all flux moving to right/left is in a trapped mode; compute numerically:

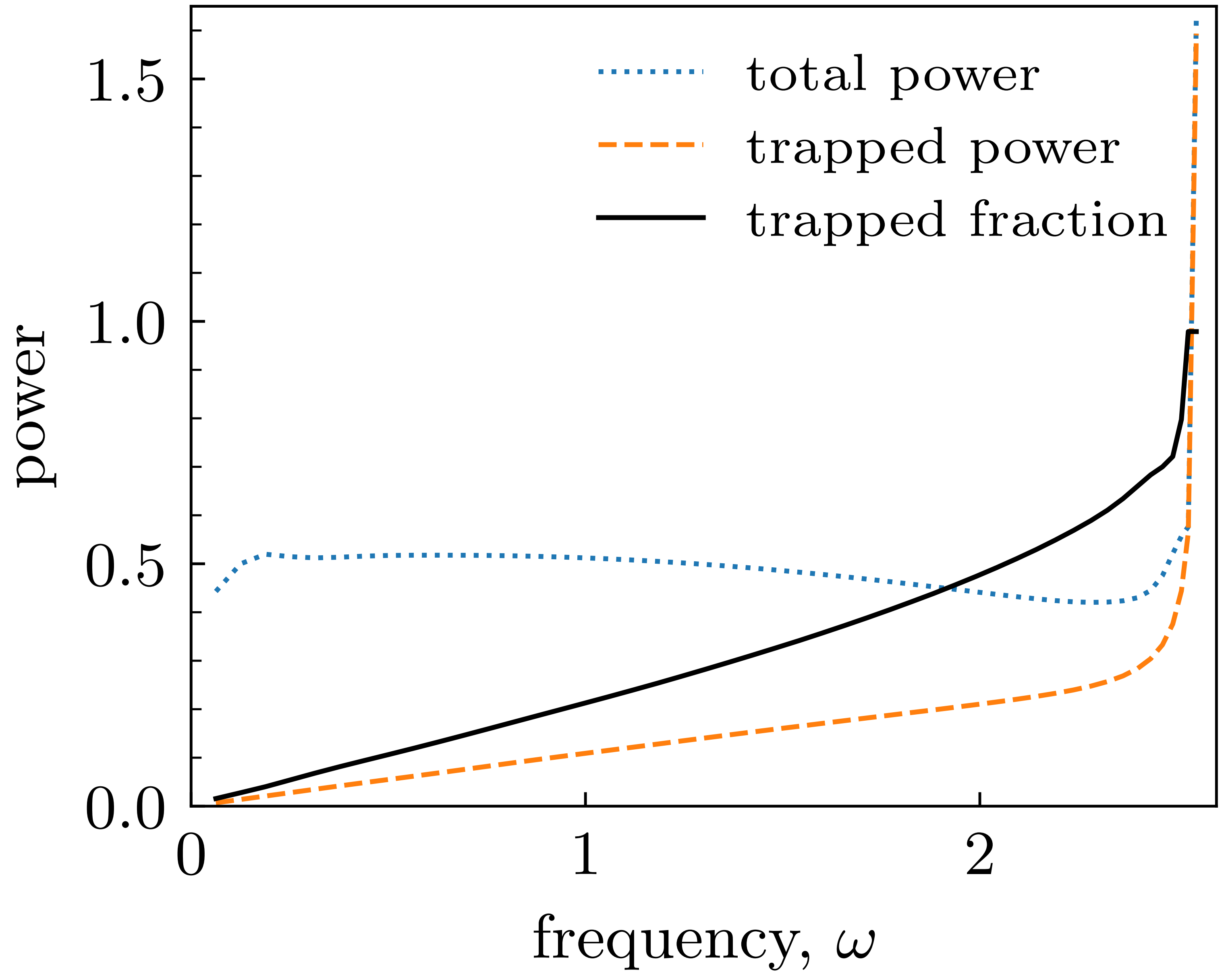
$$F_{\text{trapped},\rightarrow} = \Im\left(\int_{x_{2,0}}^{a} \bar{u}\partial_{x_1}u\,\mathrm{d}x_2\right),$$

where integral extends **from boundary** to where the mode has sufficiently decayed, but at what x_1?



- Simple Gauss—Legendre, closest node no closer than width of smallest panel on boundary.
- Total power injected into the system is $F_{\text{tot}} = \frac{1}{4} + \Im(u(\mathbf{x}_0))$, with $\mathbf{x}_0$ the source location.





Future work

- How does the position of the source affect the power distribution in trapped modes?
  - Can a left/right asymmetry be induced?
  - What happens in asymmetric geometries?
  - Can we derive an fast, approximate model for the power distribution for applications such as nondestructive sensing?
- Poles coalesce as $\kappa \to 0, \pm \pi$, more quadrature nodes and differently shaped path needed in array scanning integral to preserve accuracy
- 3D periodic surfaces: band structure complex, poles are lines
- Inverse problem for fault detection in periodic structures (e.g. photonic crystals)


Thank you

Periodization II — Wood anomalies

- At κ-values where k_n^2 changes sign, i.e. $\kappa + 2n\pi/d = \pm\,\omega$
  - Behavior of periodic Green's function in the x_2 direction changes: **oscillatory → evanescent**
  - Quasiperiodic Green's function does not exist (!)
- Criss-cross lines in $\omega - \kappa$ plane
- Due to symmetry, we can restrict ourselves to the first **Brillouin zone** (shown in red)