Reproducing the unique acoustics of periodic staircases using boundary integral equations

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Acoustics from corrugated surfaces in real life

Motivation and goals

- Interesting acoustic phenomena near corrugated surfaces, e.g. step-temples:

El Castillo (“The Castle”), a Mesoamerican step-pyramid in Chichen Itza, Mexico.

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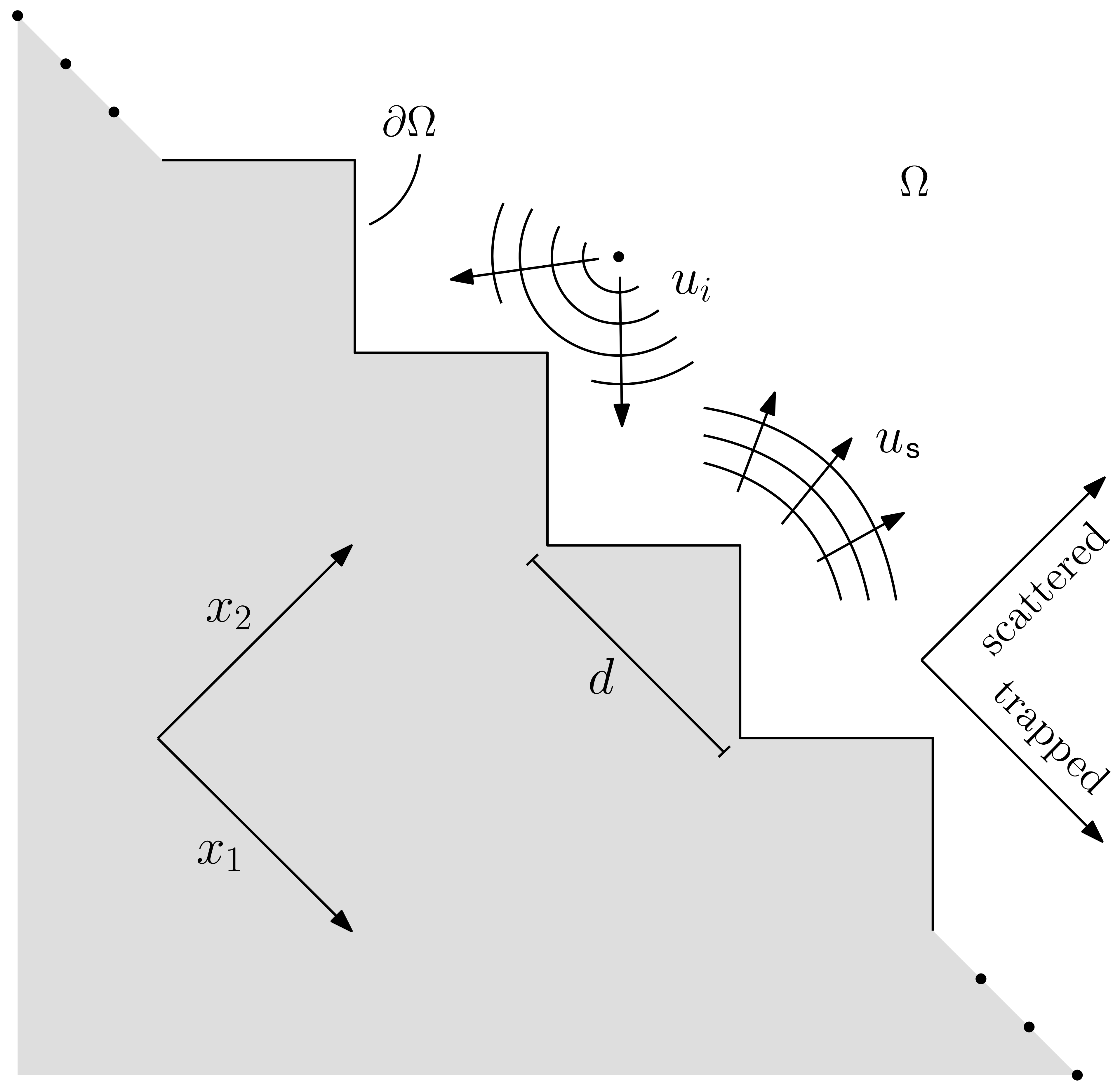
- Use the **array scanning method** to arrive at the solution from a **single point source,** from periodic array of point sources
  - This will involve integrating over the **quasiperiodicity parameter,** κ

Problem setup - multiple sources

$(\Delta + \omega^2)u = 0$	in Ω,	PDE
$u_n = 0$	on $\partial\Omega$,	boundary condition (Neumann)
$u(x_1 + nd, x_2) = \alpha^n u(x_1, x_2)$	$(x_1, x_2) \in \Omega$,	quasiperiodicity
$u(x_1, x_2) = \sum_{n\in\mathbb{Z}} c_n e^{i(\kappa_n x_1 + k_n x_2)}$,	$x_2 > x_2^{(0)}$	radiation condition



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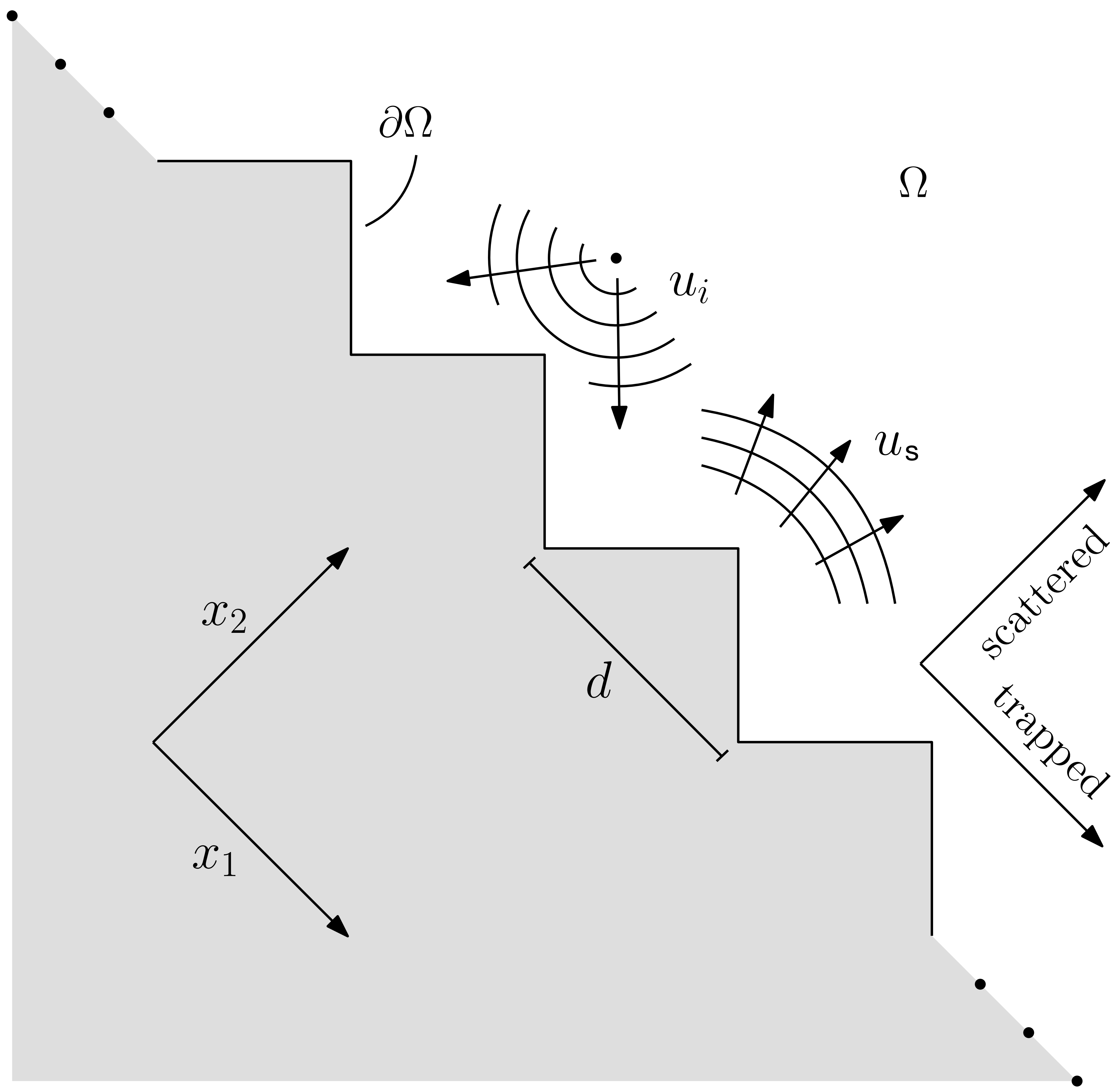

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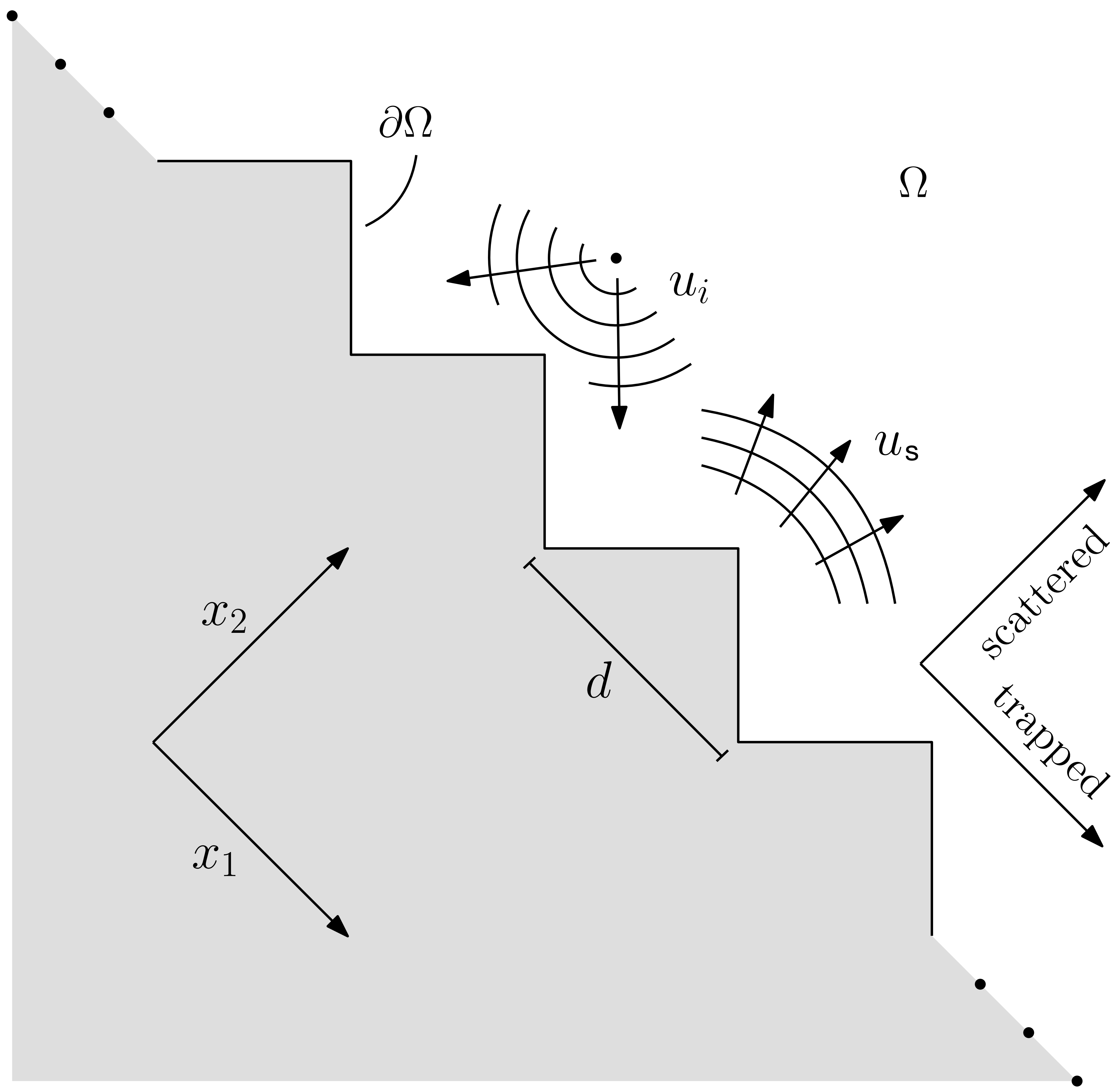

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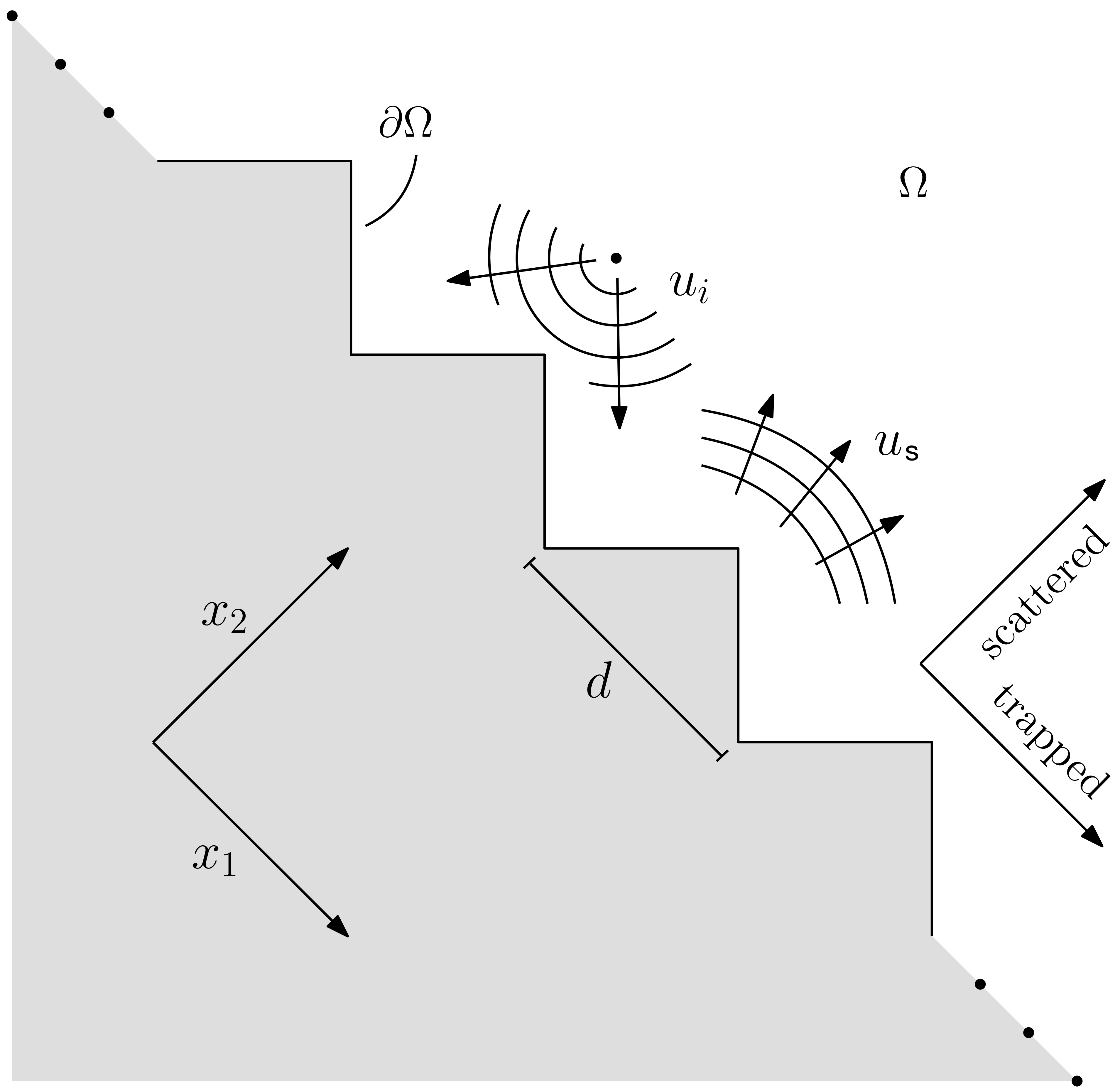

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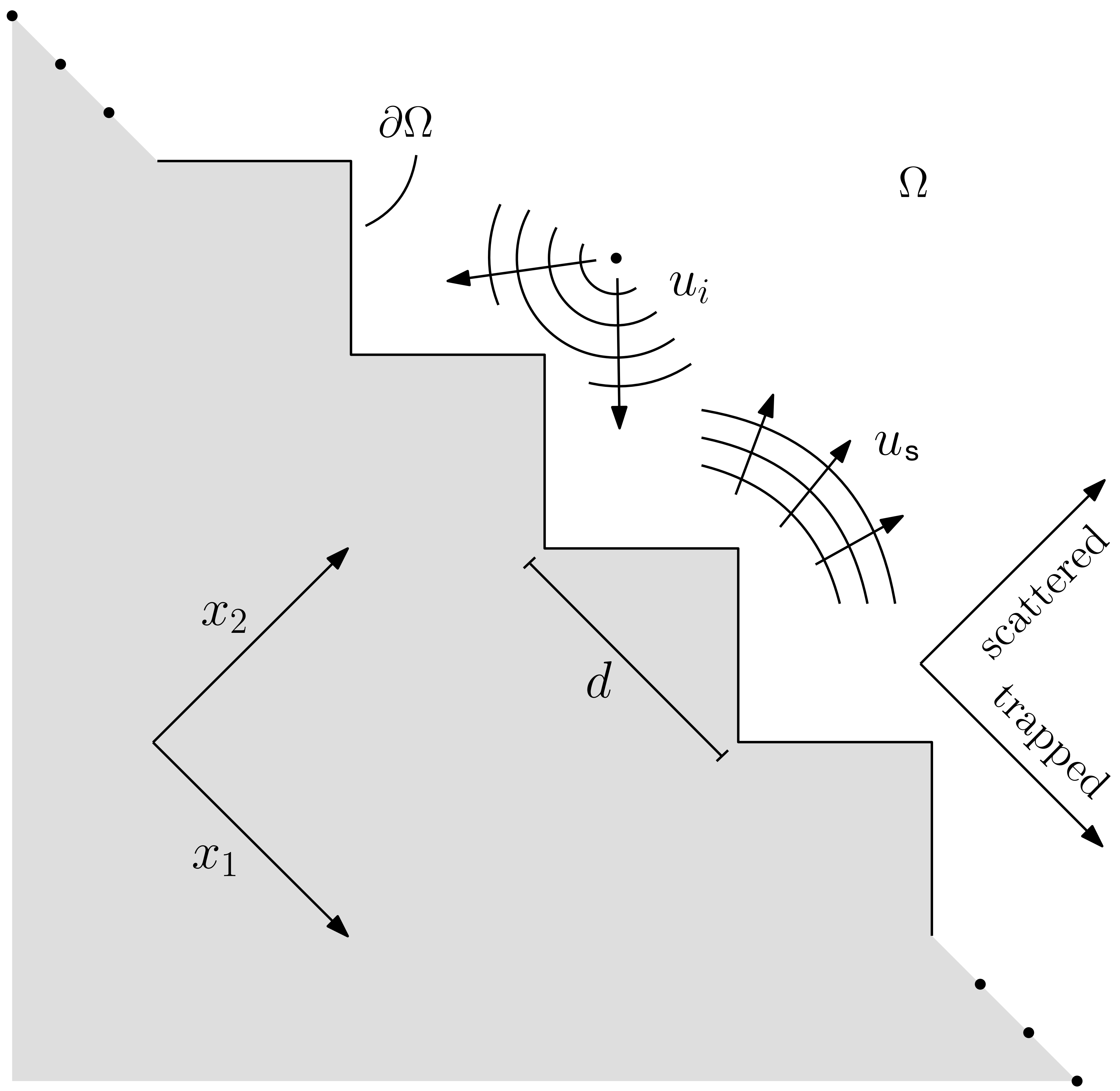

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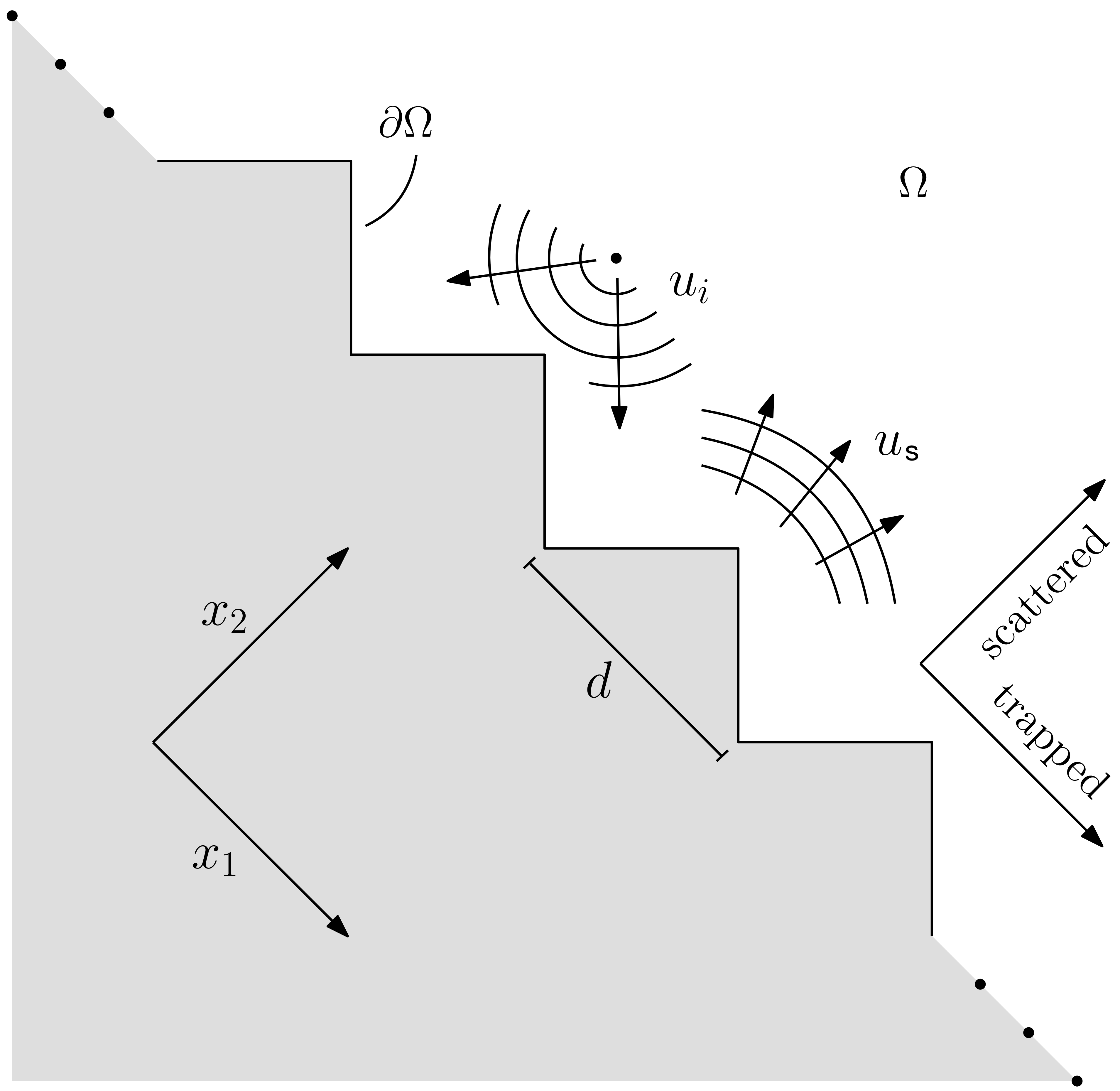
- If the total wavevector is $\mathbf{k} = (\kappa_n, k_n)$, then $k_n = \sqrt{\omega^2 - \kappa_n^2}$ is the vertical wavevector (imaginary part always +ve)



Periodization

- Reduce computation to the unit cell by using the **periodic Green's function**, $\Phi_{\mathrm{p}}(\mathbf{x}, \mathbf{y})$, where $\mathbf{x}$ is the target's, $\mathbf{y}$ is the source's position vector:

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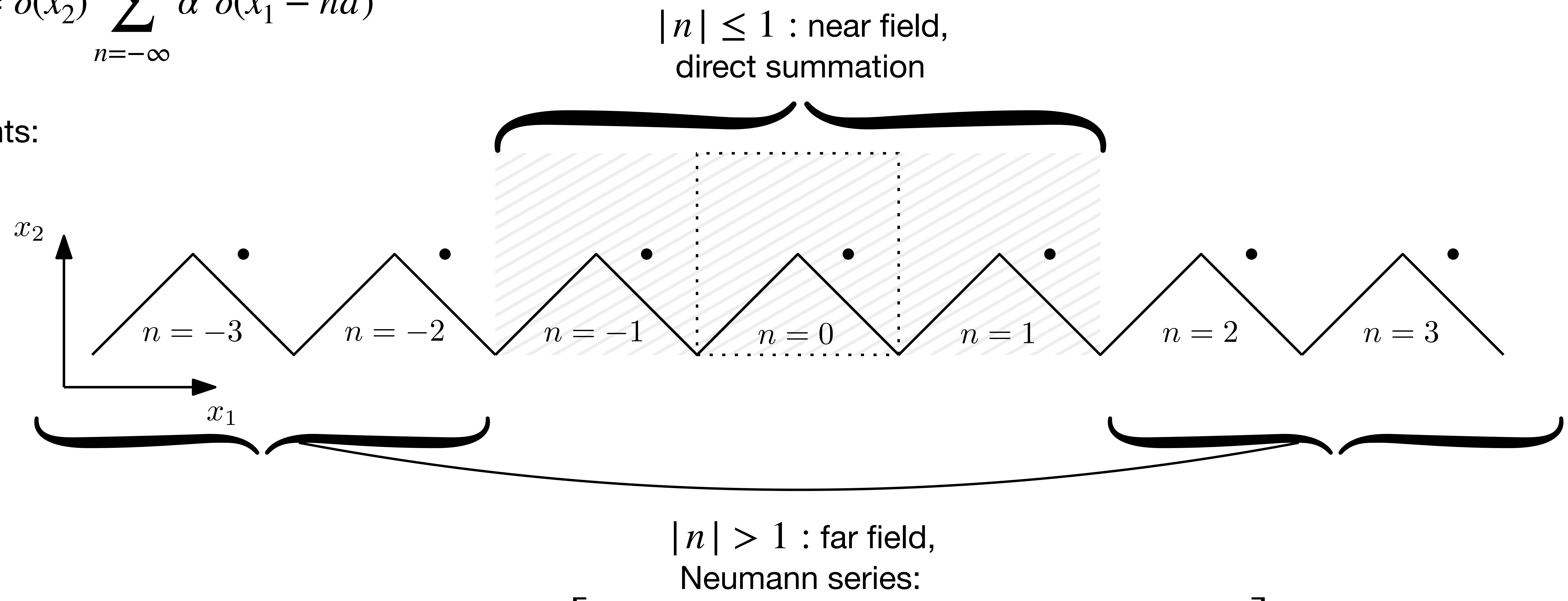

$$\Phi_{p,\mathrm{far}}(\vec{x},0) = \frac{i}{4}\left[S_0(\omega,\kappa)J_0(\omega,\vec{x}) + 2\sum_{n=1}^{\infty} S_n(\omega,\kappa)J_n(\omega,\vec{x})a(\vec{x})\right]$$

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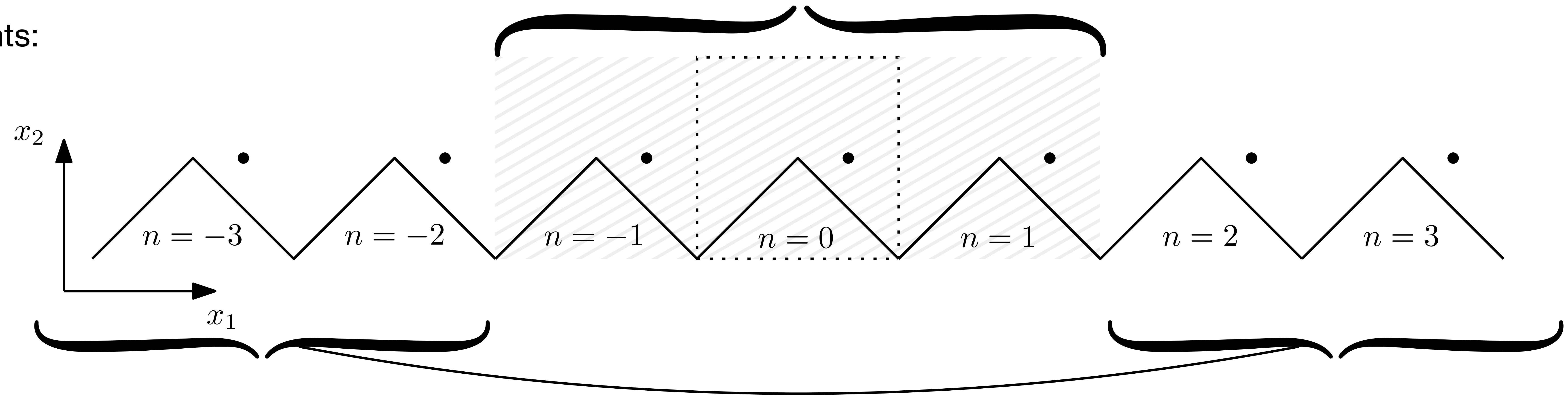
$|n| \leq 1$: near field, direct summation



$|n| > 1$: far field, Neumann series:

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- The $S_n(\omega, \kappa)$ are **lattice sums** involving sums over n-th order Hankel functions
  - Computed once per ω, κ
  - Slowly convergent → use integral representation (Yasumoto and Yoshitomi, IEEETAP, 1999)
  - Only valid inside unit cell

Periodization II — Wood anomalies

- At κ-values where k_n^2 changes sign, i.e. $\kappa + 2n\pi = \pm\,\omega$
  - Behavior of periodic Green's function in the x_2 direction changes:

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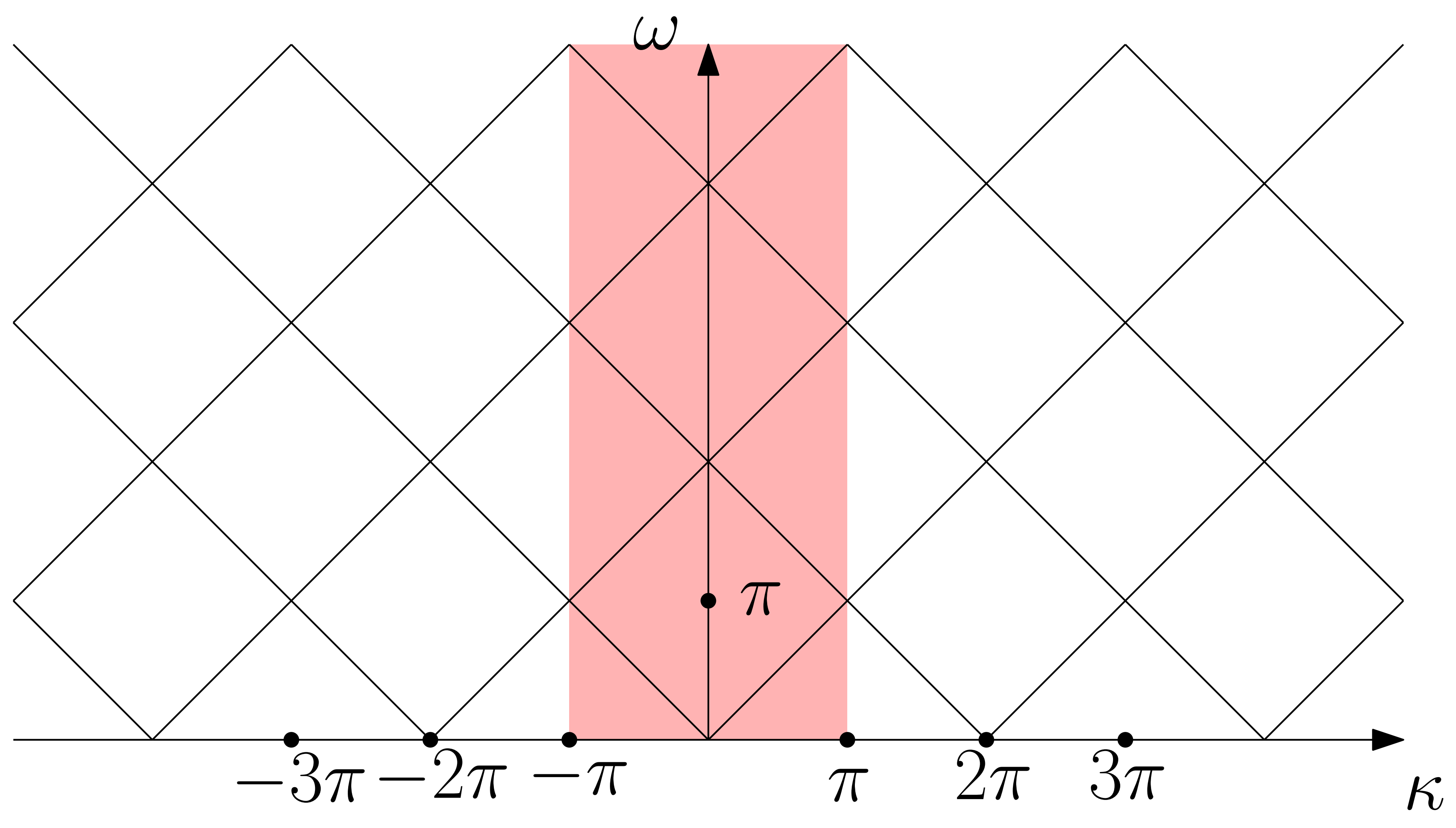

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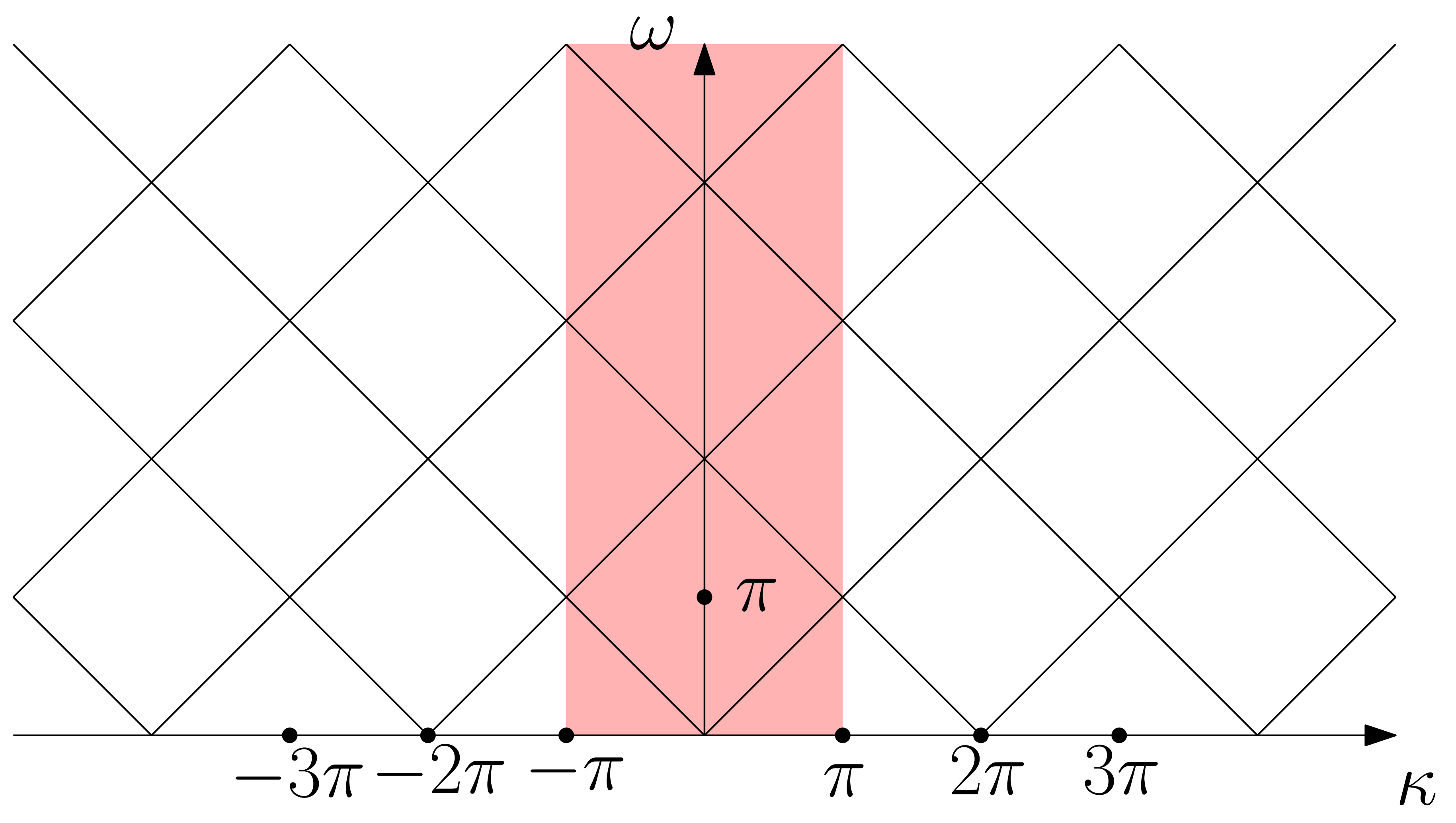
- Due to symmetry, we can restrict ourselves to the first **Brillouin zone** (shown in red)



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D. Colton and R. Kress, *Inverse Acoustic and Electromagnetic Scattering Theory*
D. Colton and R. Kress, *Integral Equation Methods in Scattering Theory*
R. Kress, *Linear Integral Equations*
I. Stakgold, *Boundary value problems of mathematical physics*,
Paul Garabedian, *Partial Differential Equations*

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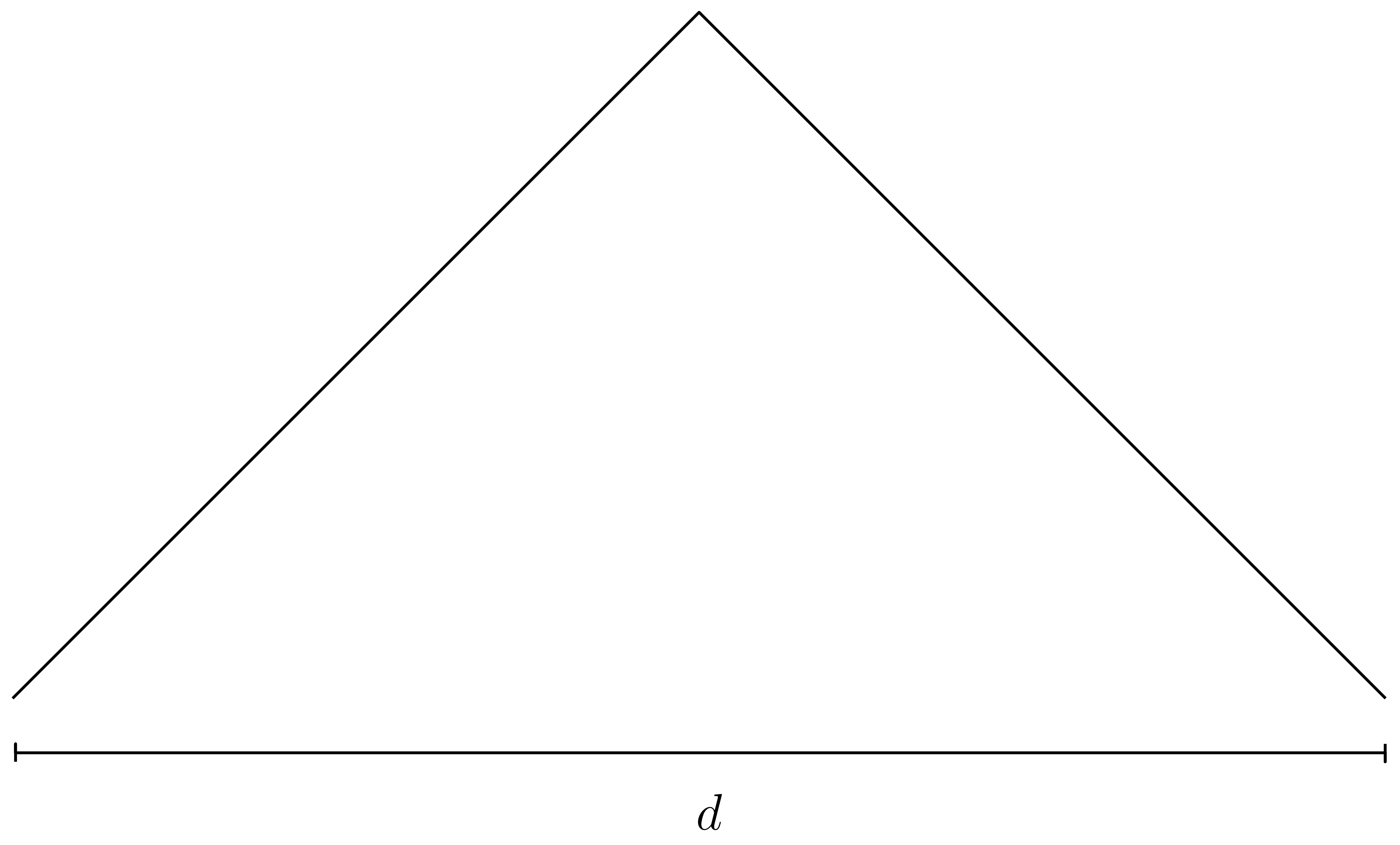

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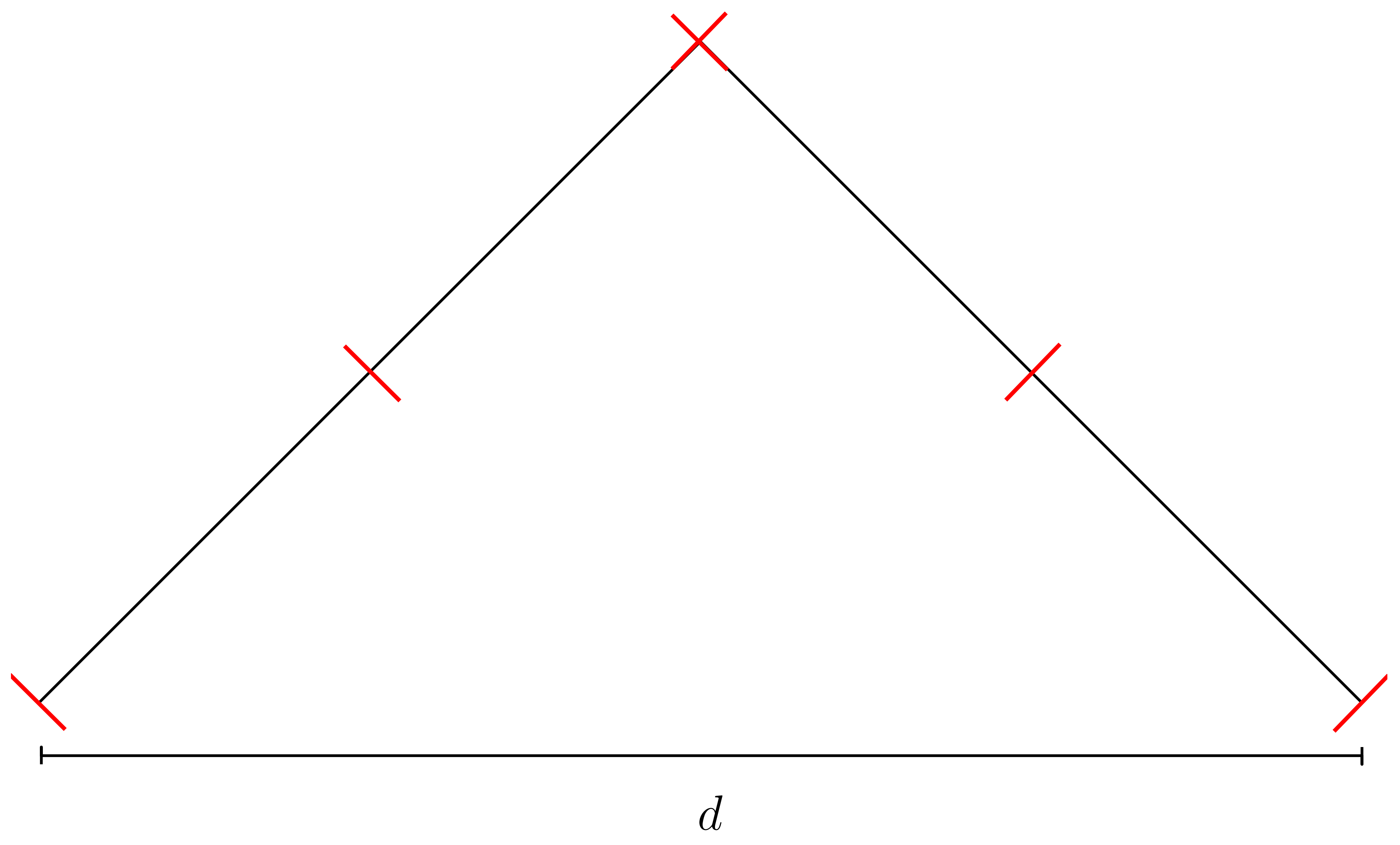

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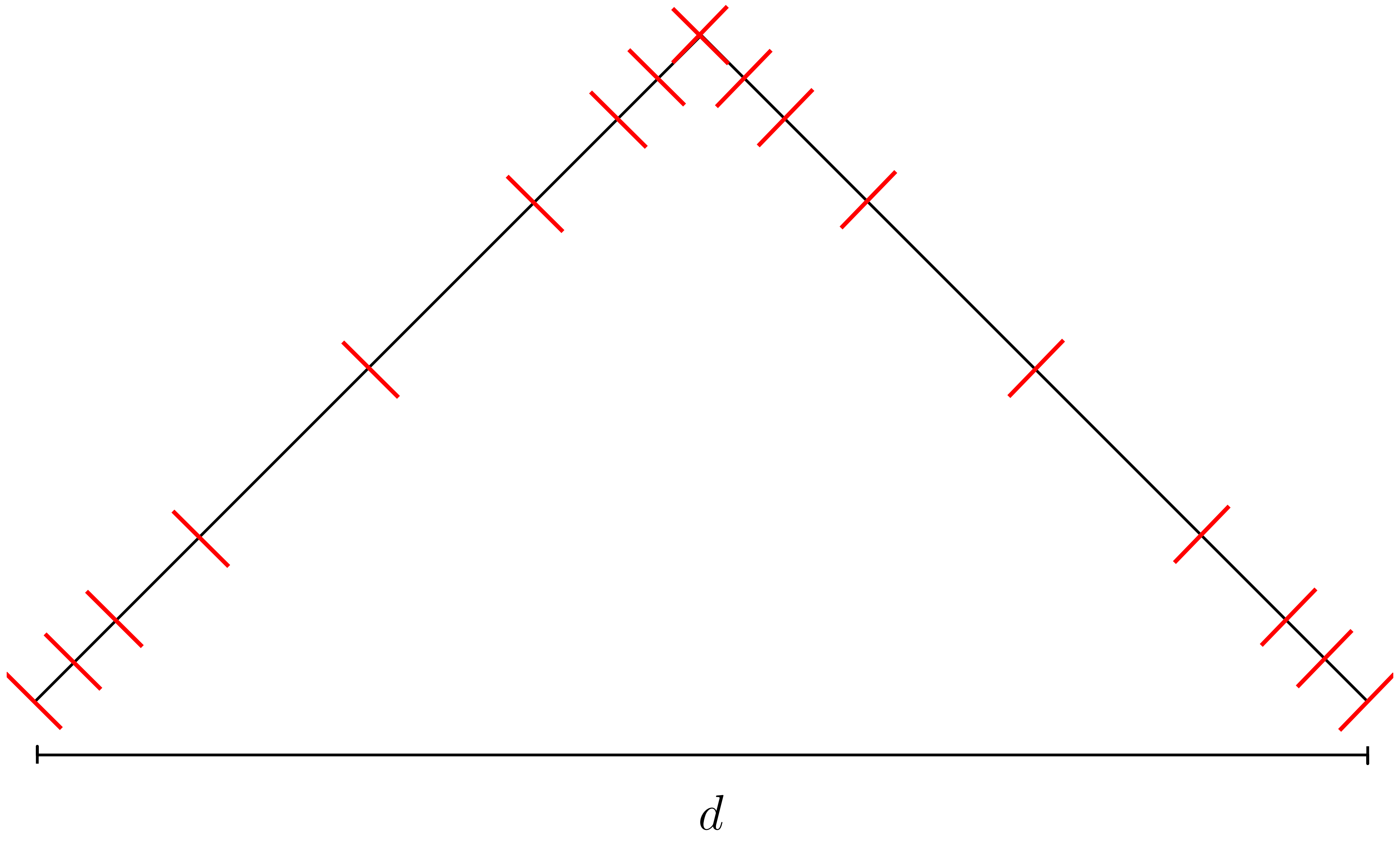

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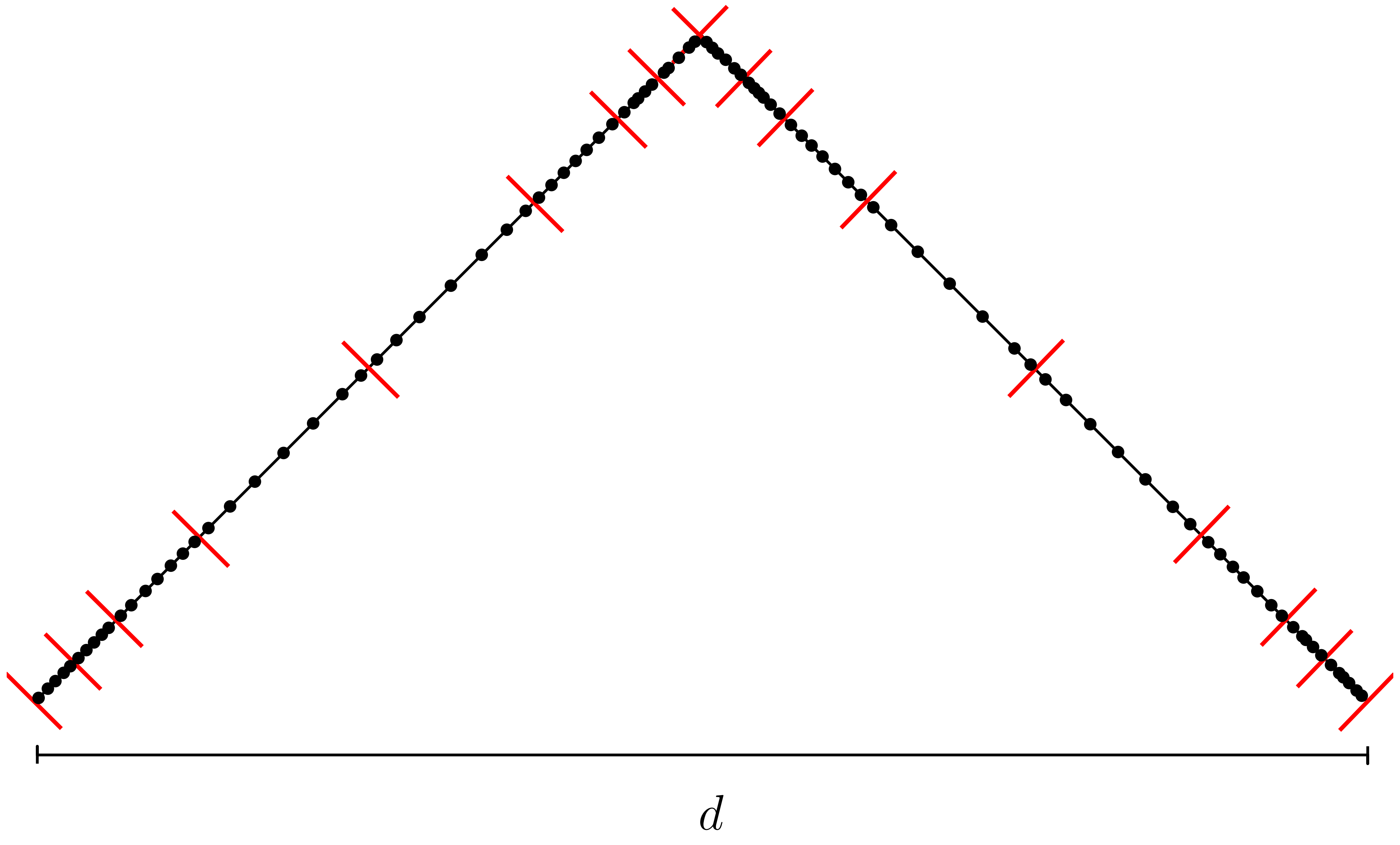

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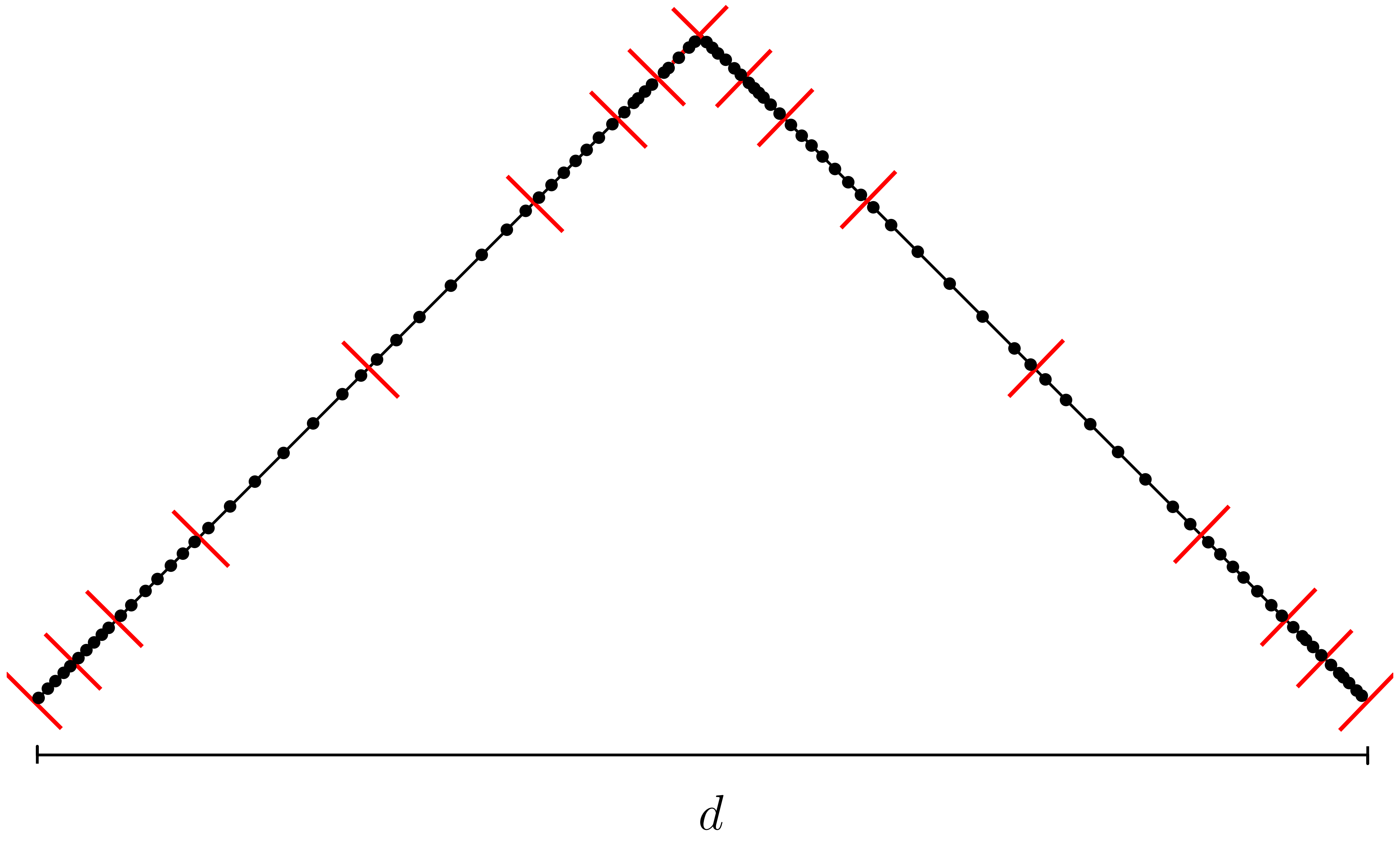
- Quadrature coordinates **relative** to the nearest corner to avoid catastrophic cancellation
- No special rules (yet) for close evaluation



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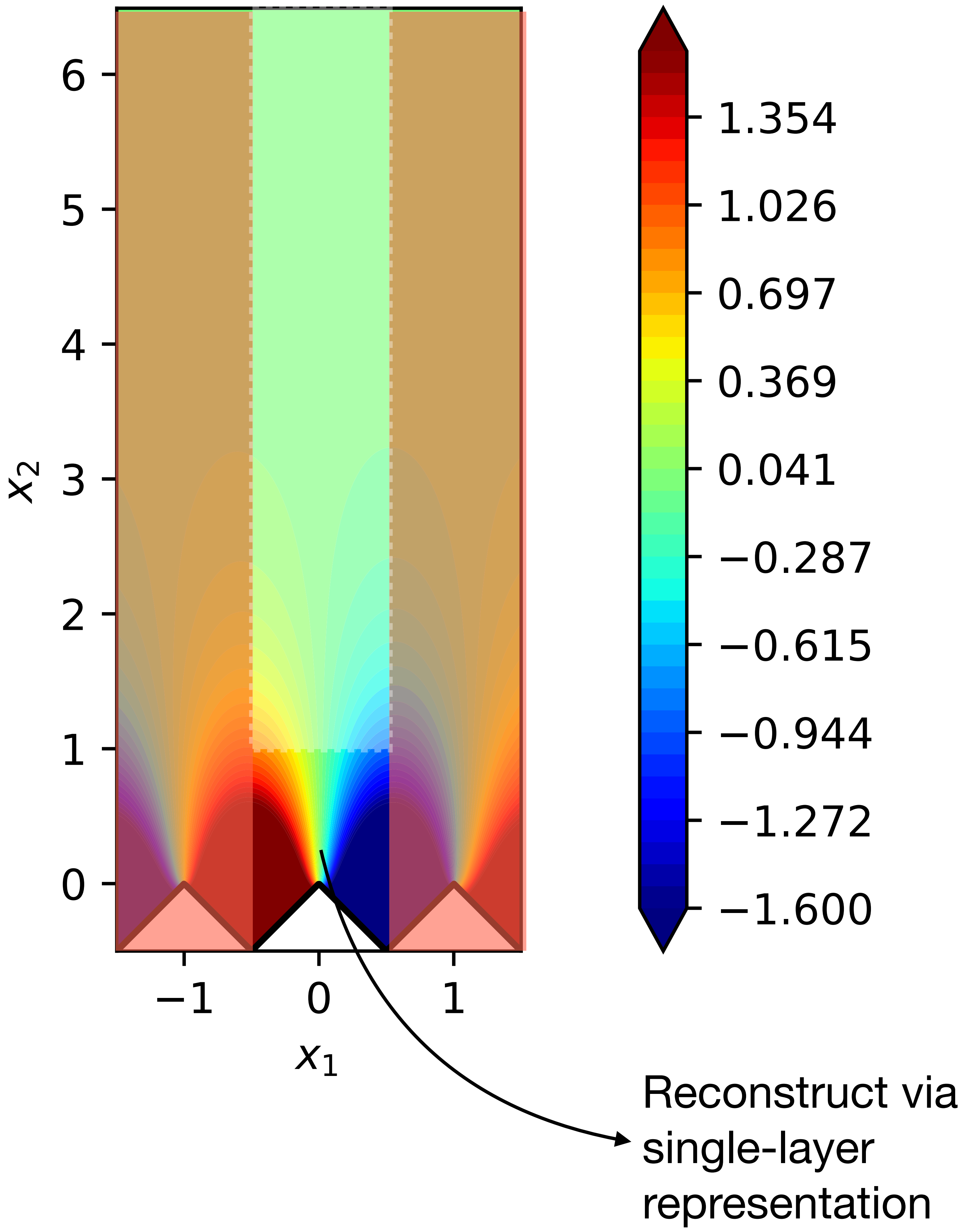

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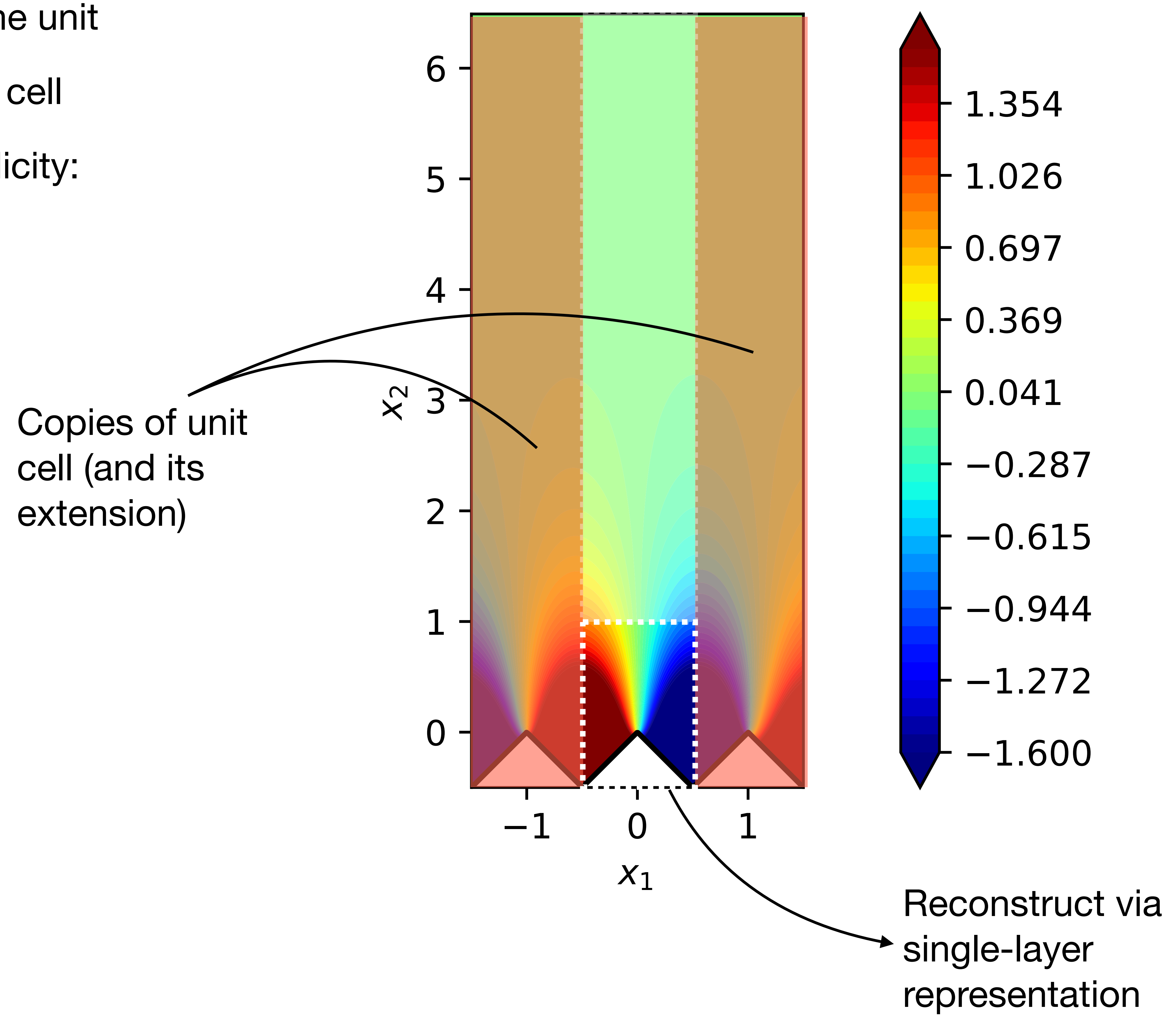

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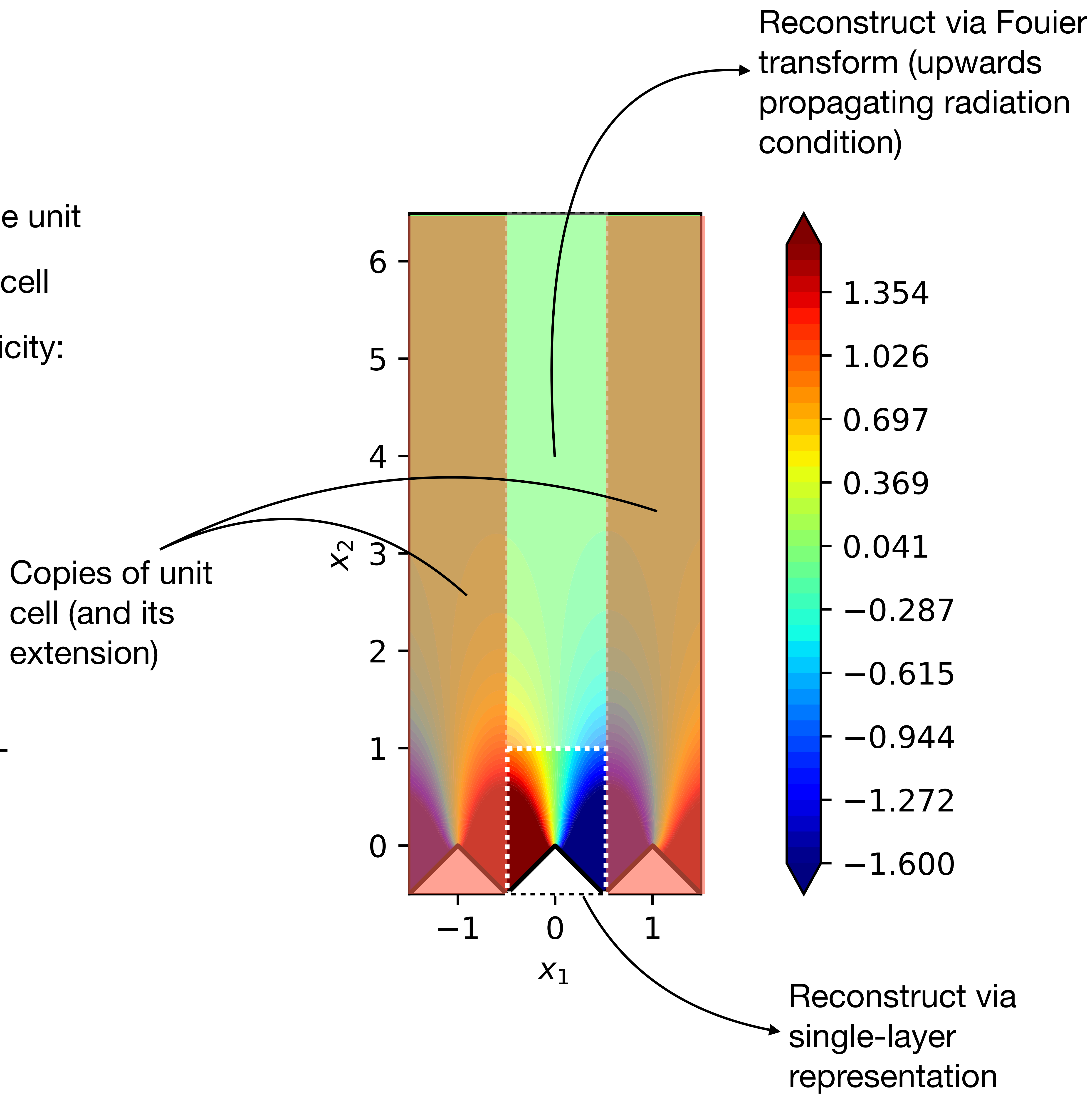

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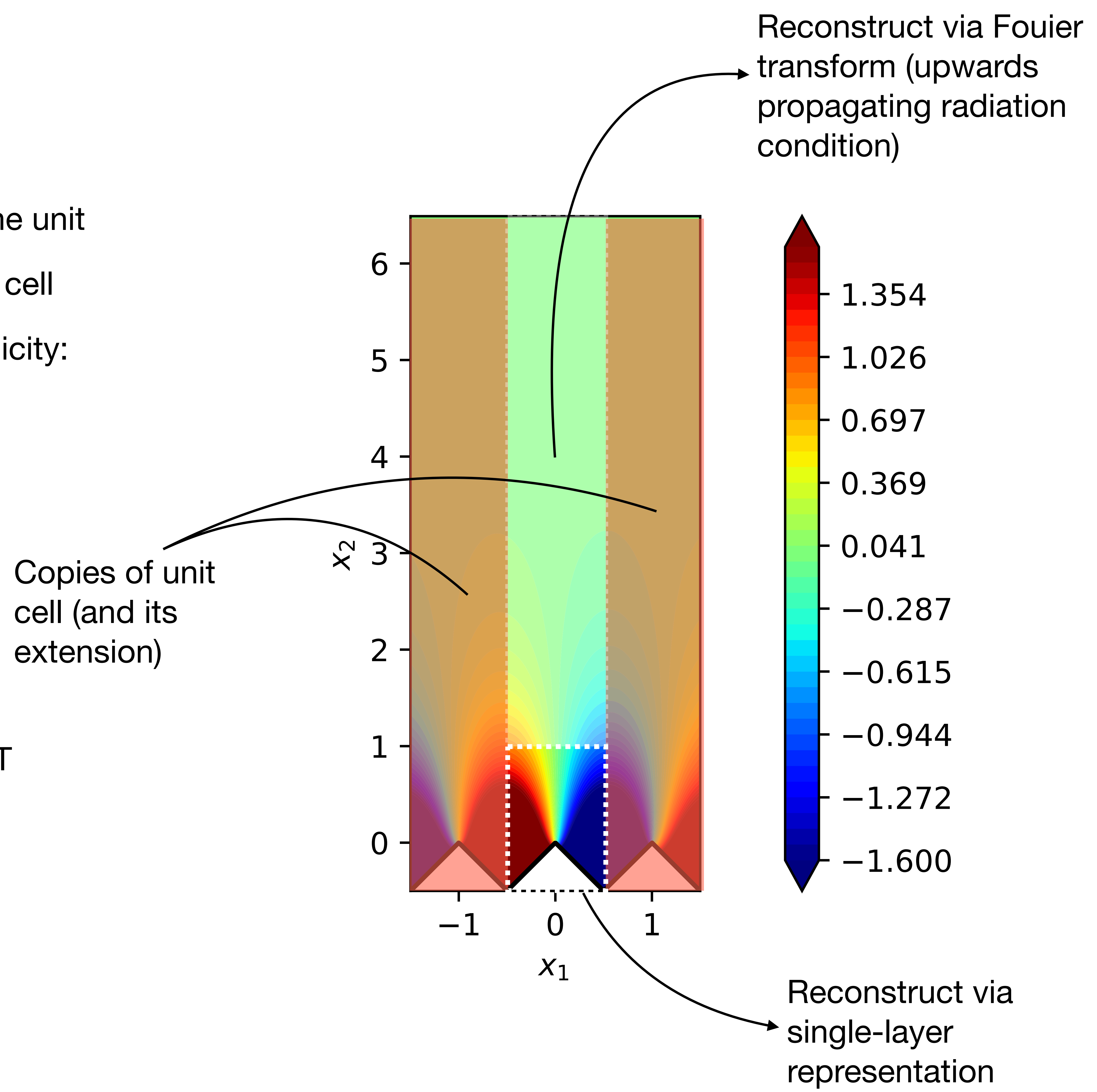

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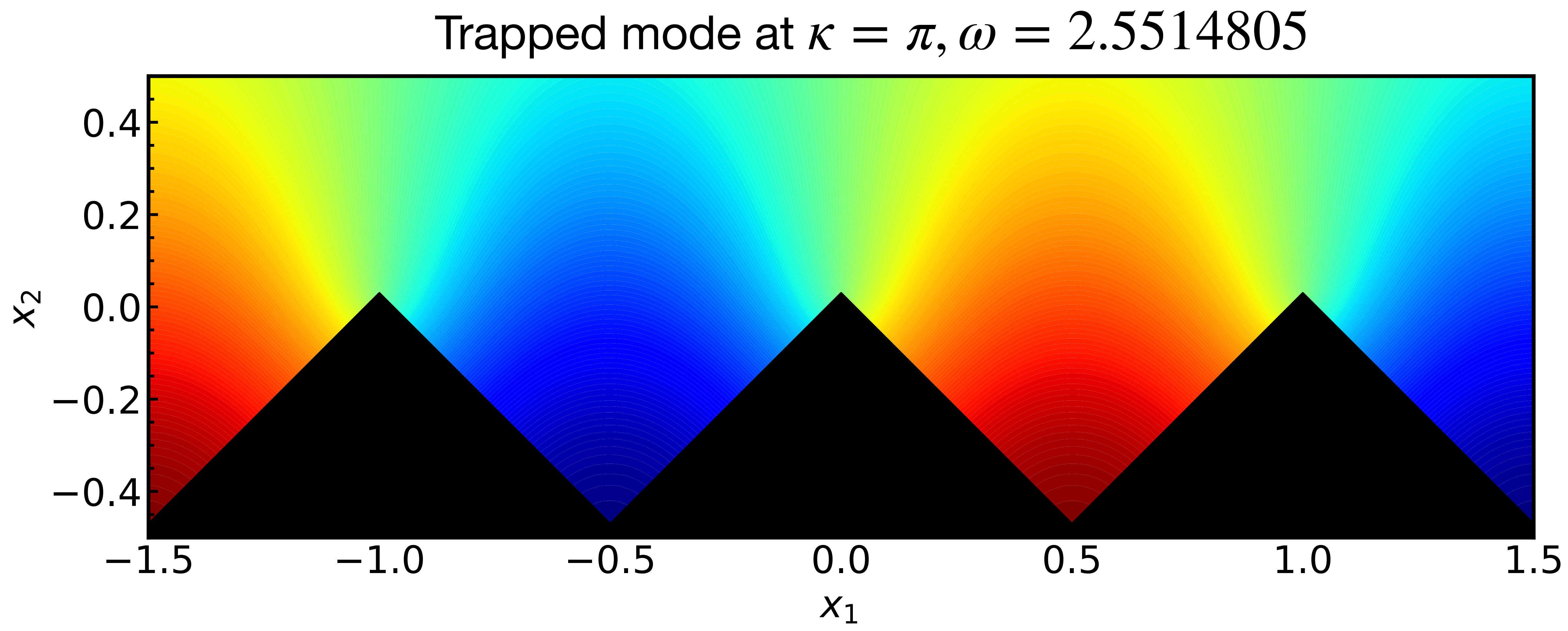

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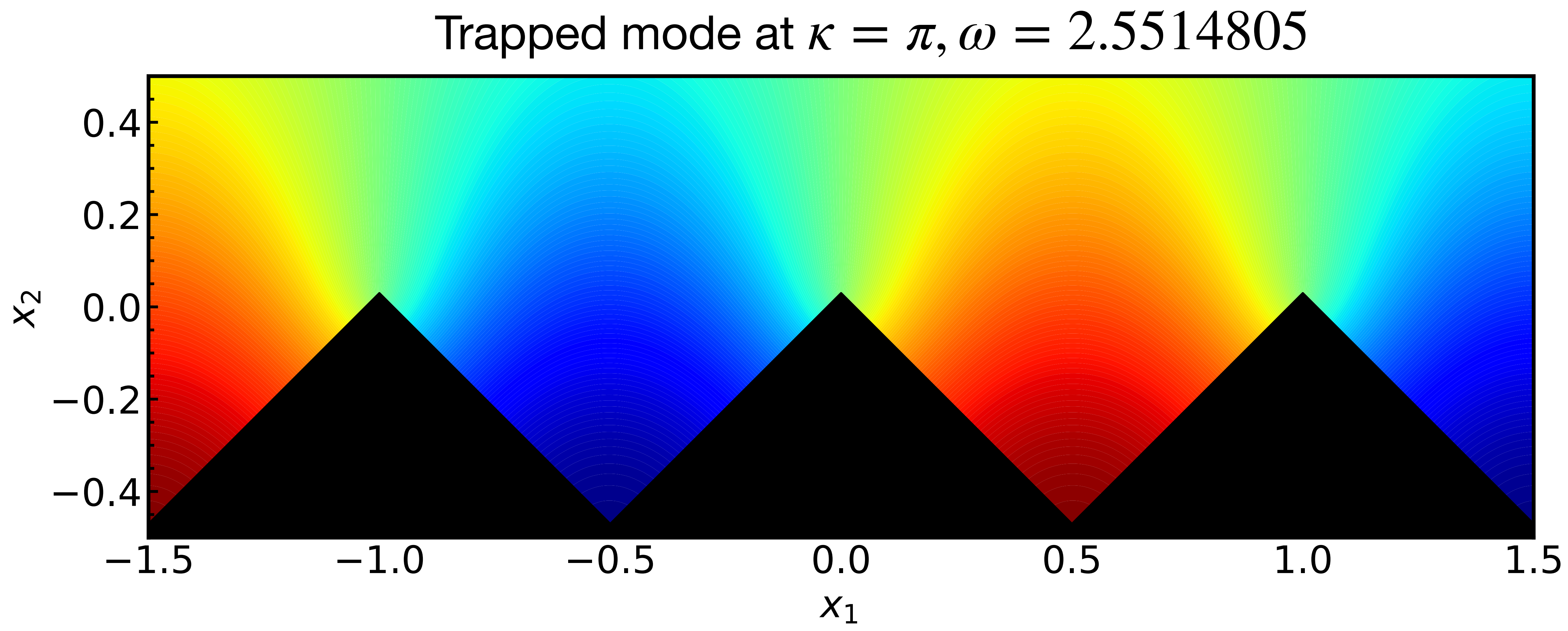

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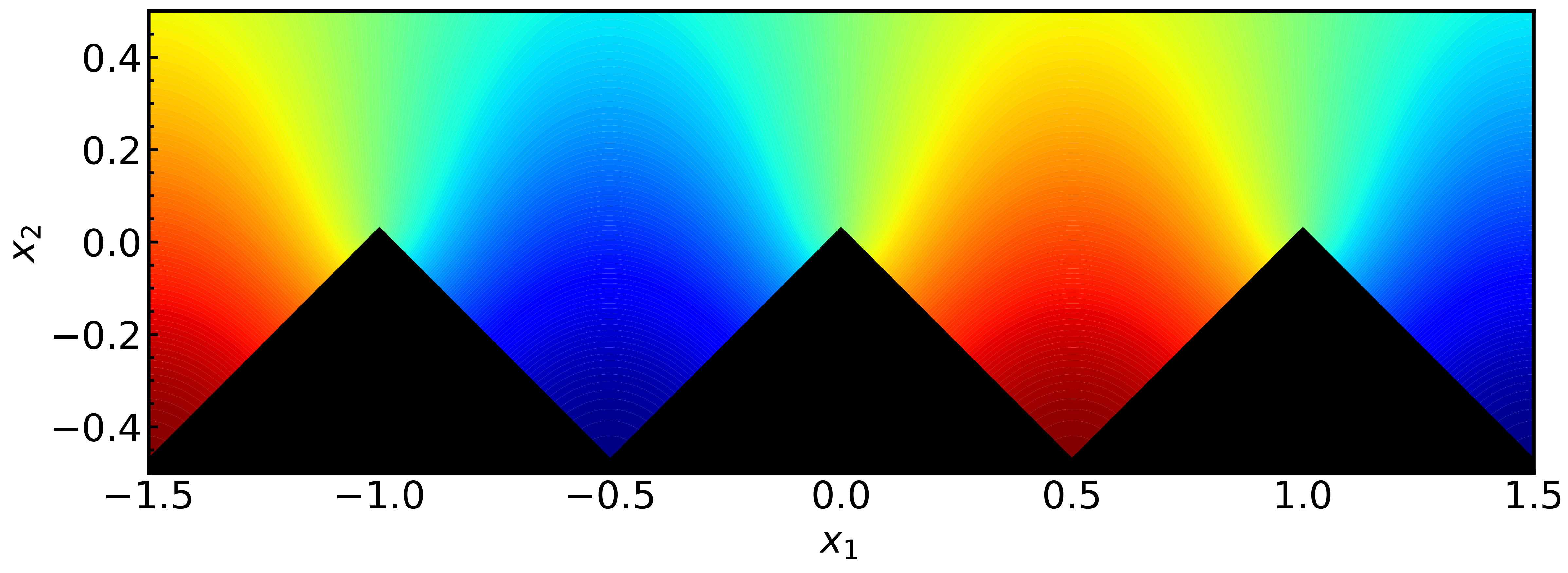
Trapped mode at $\kappa = \pi, \omega = 2.5514805$



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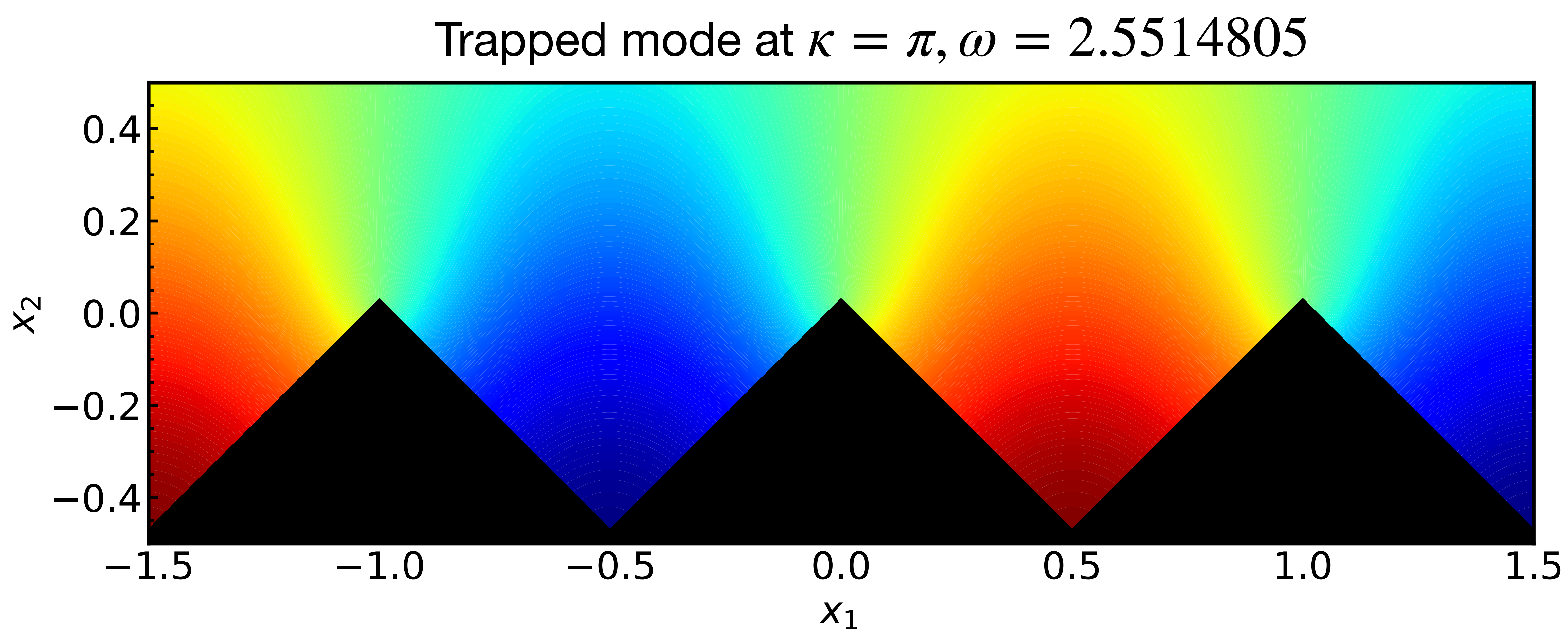
  - The group velocity of a trapped mode, $\frac{\mathrm{d}\omega}{\mathrm{d}\kappa}$, velocity at which the envelope of a wavepacket travels
  - Simple ray model: arrival time of different frequencies



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1. **Dispersion relation**

- For Neumann boundary data, there exists a trapped mode at every κ
- As $\kappa \to 0$, approaches **light line** $\omega = \kappa$
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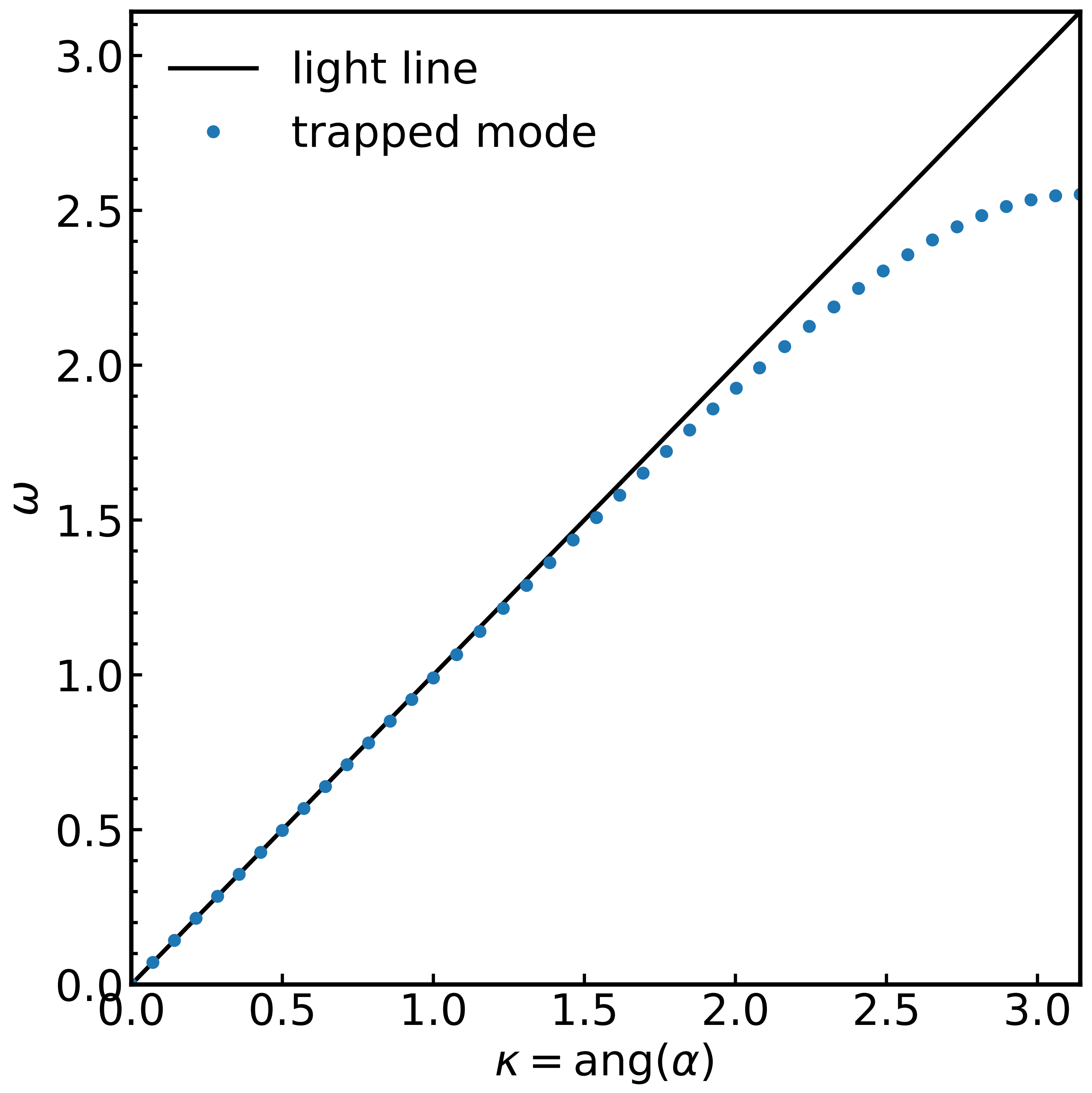

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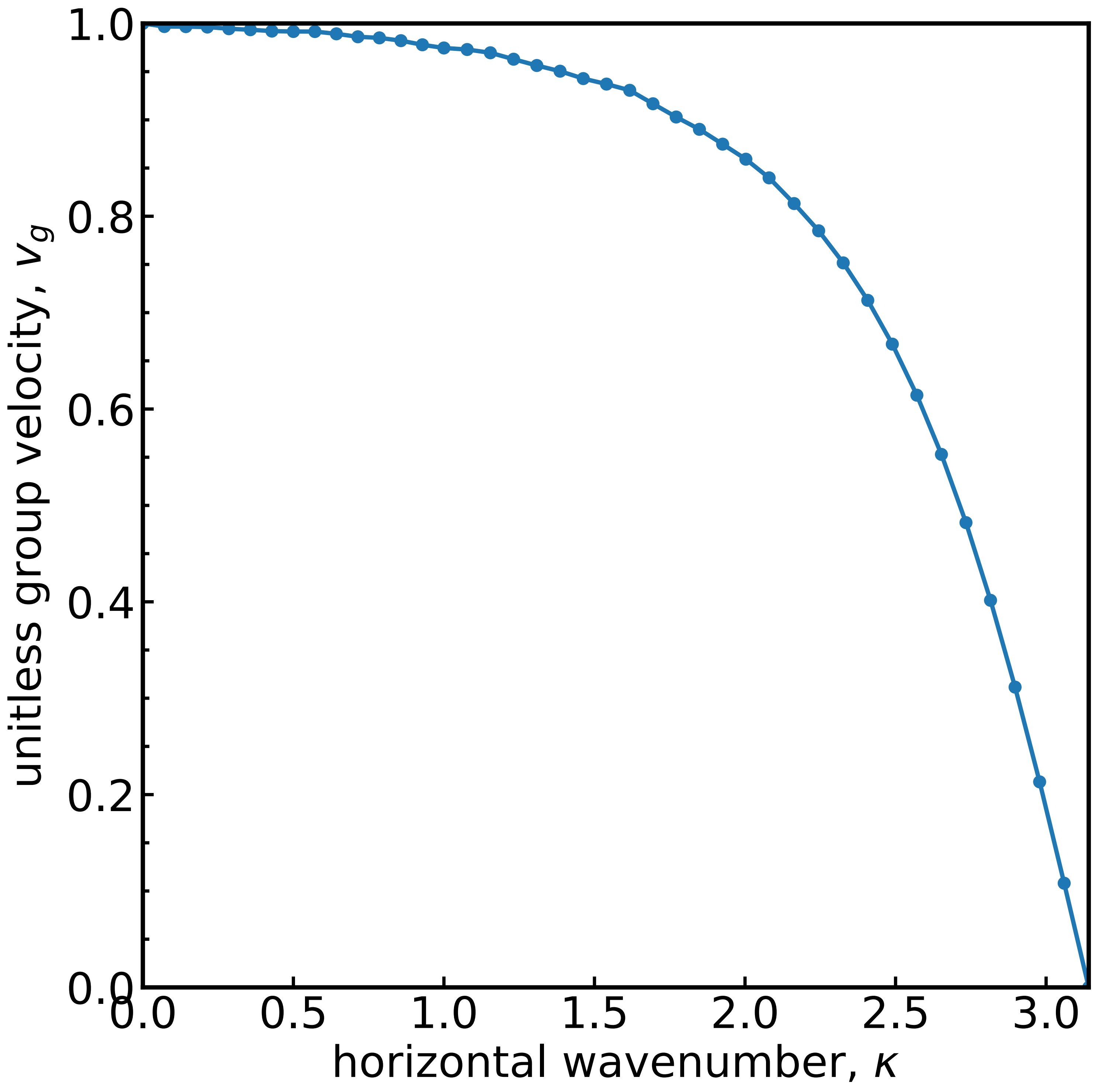

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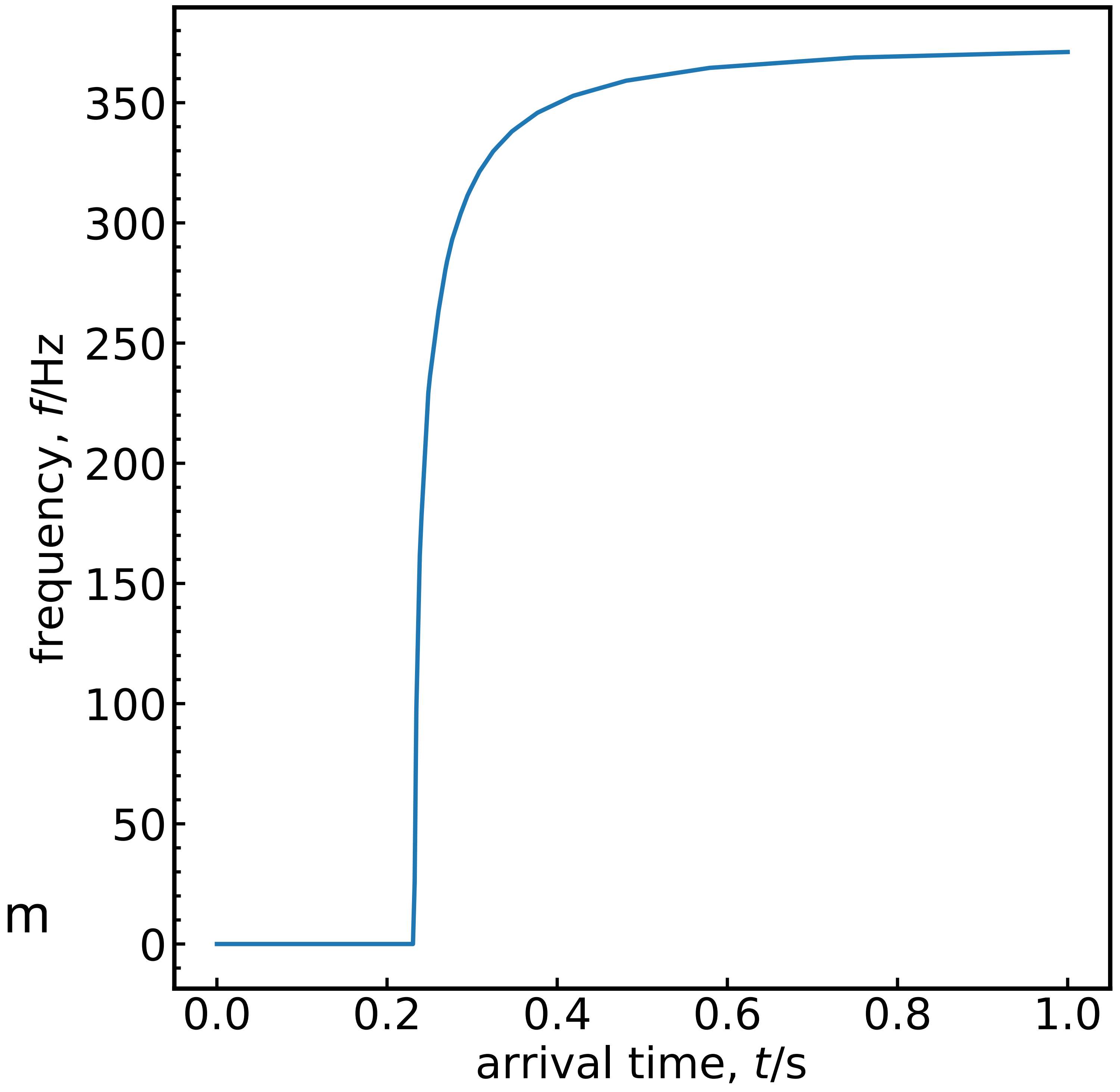

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then a single point source is

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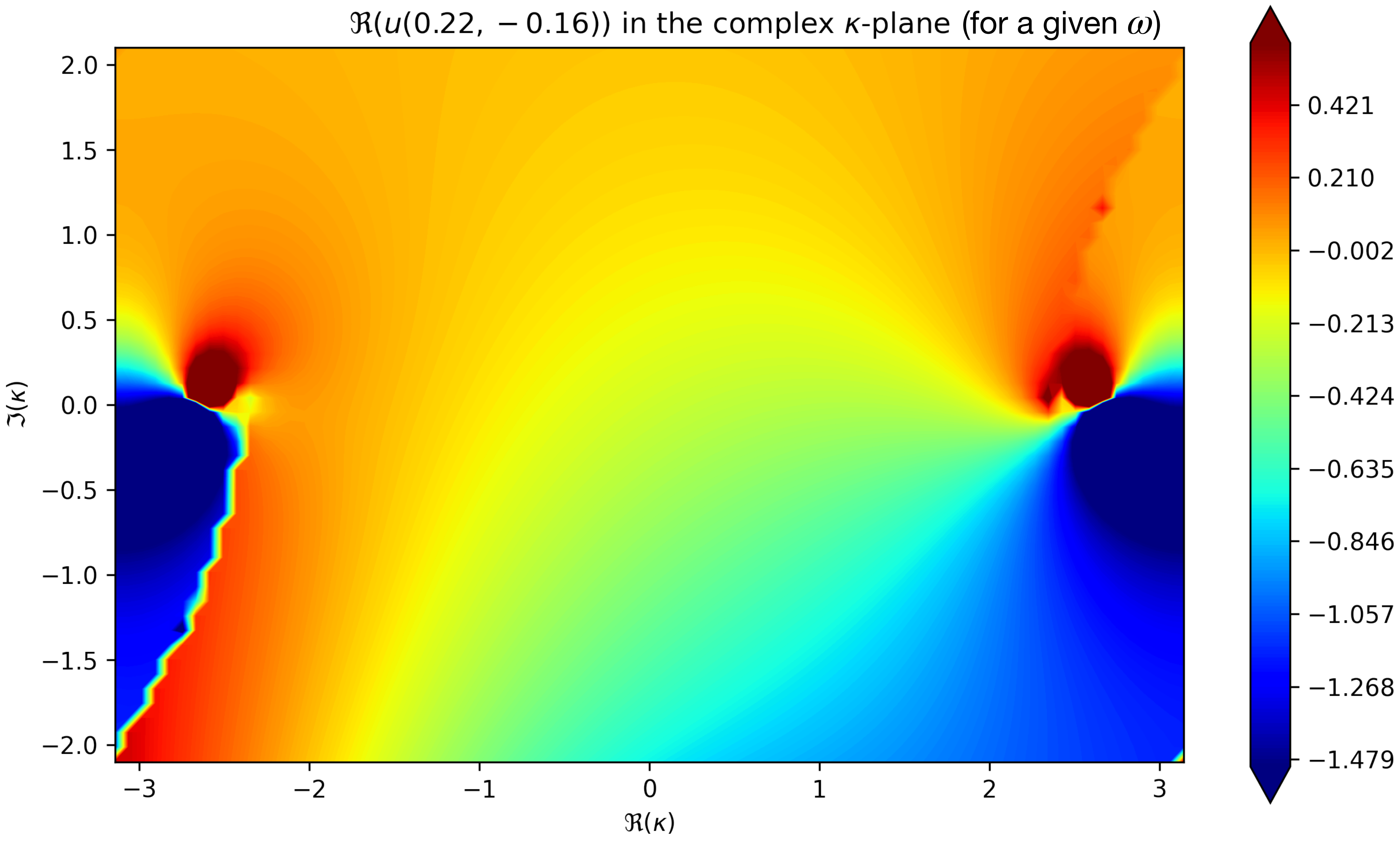

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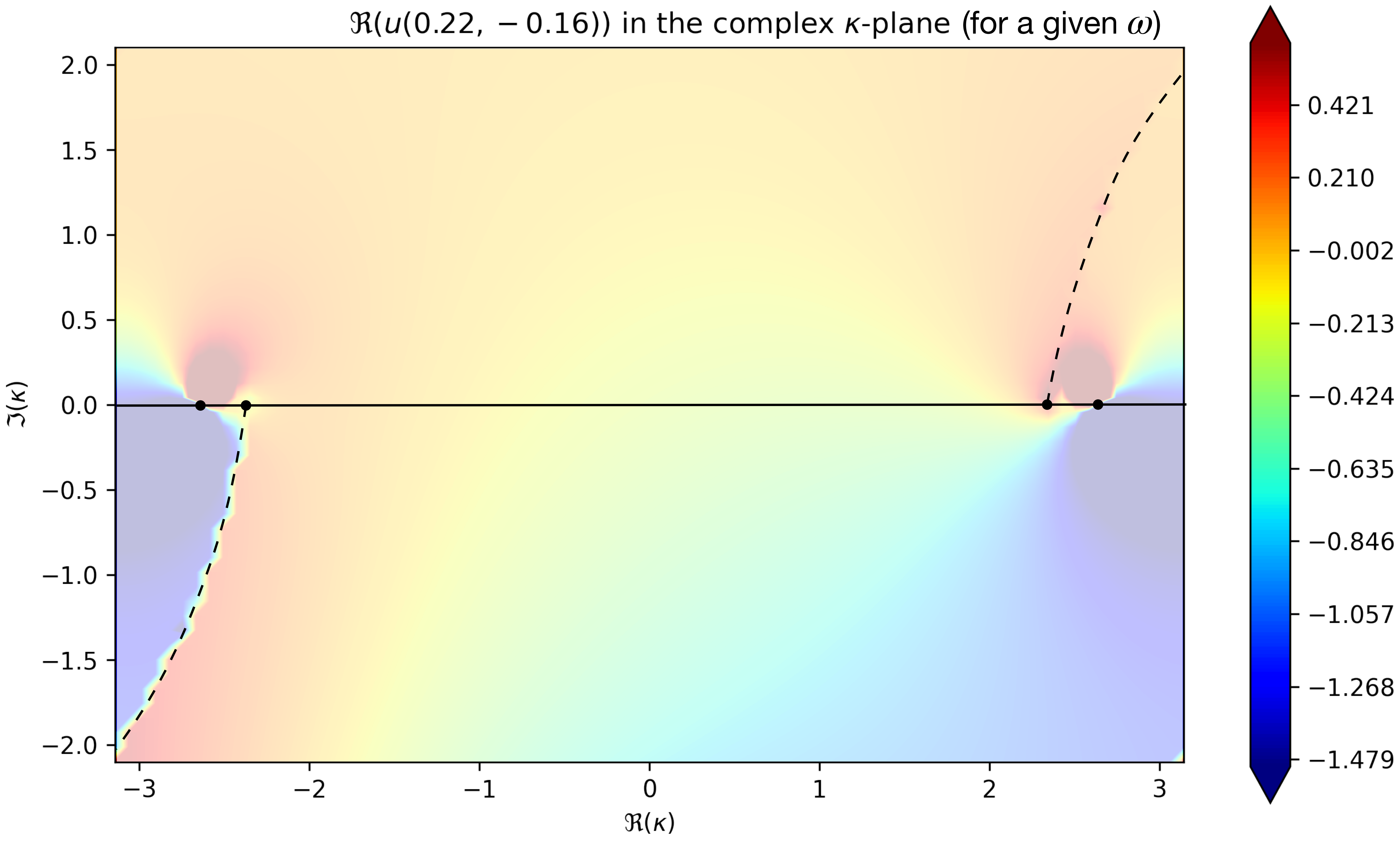

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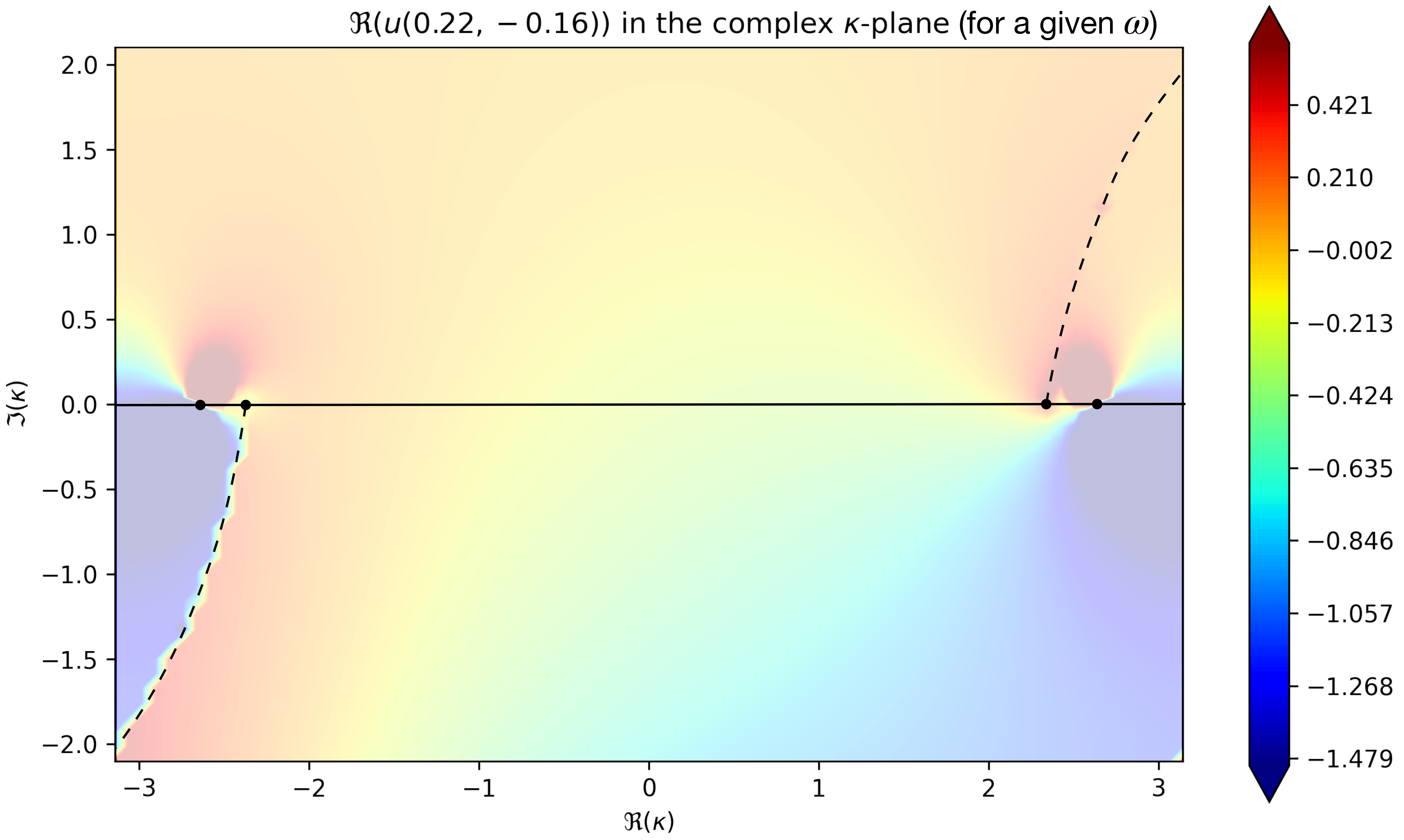

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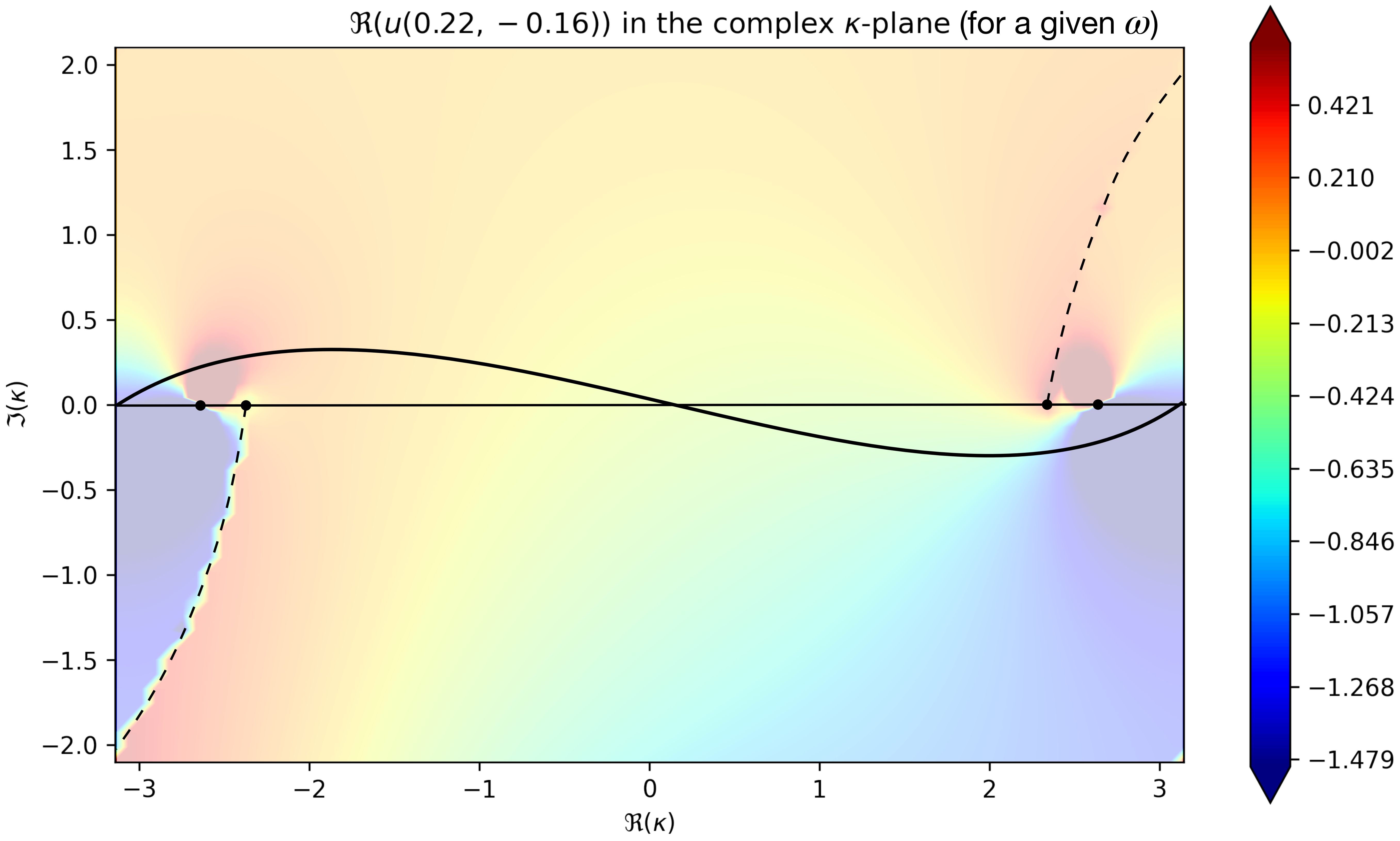

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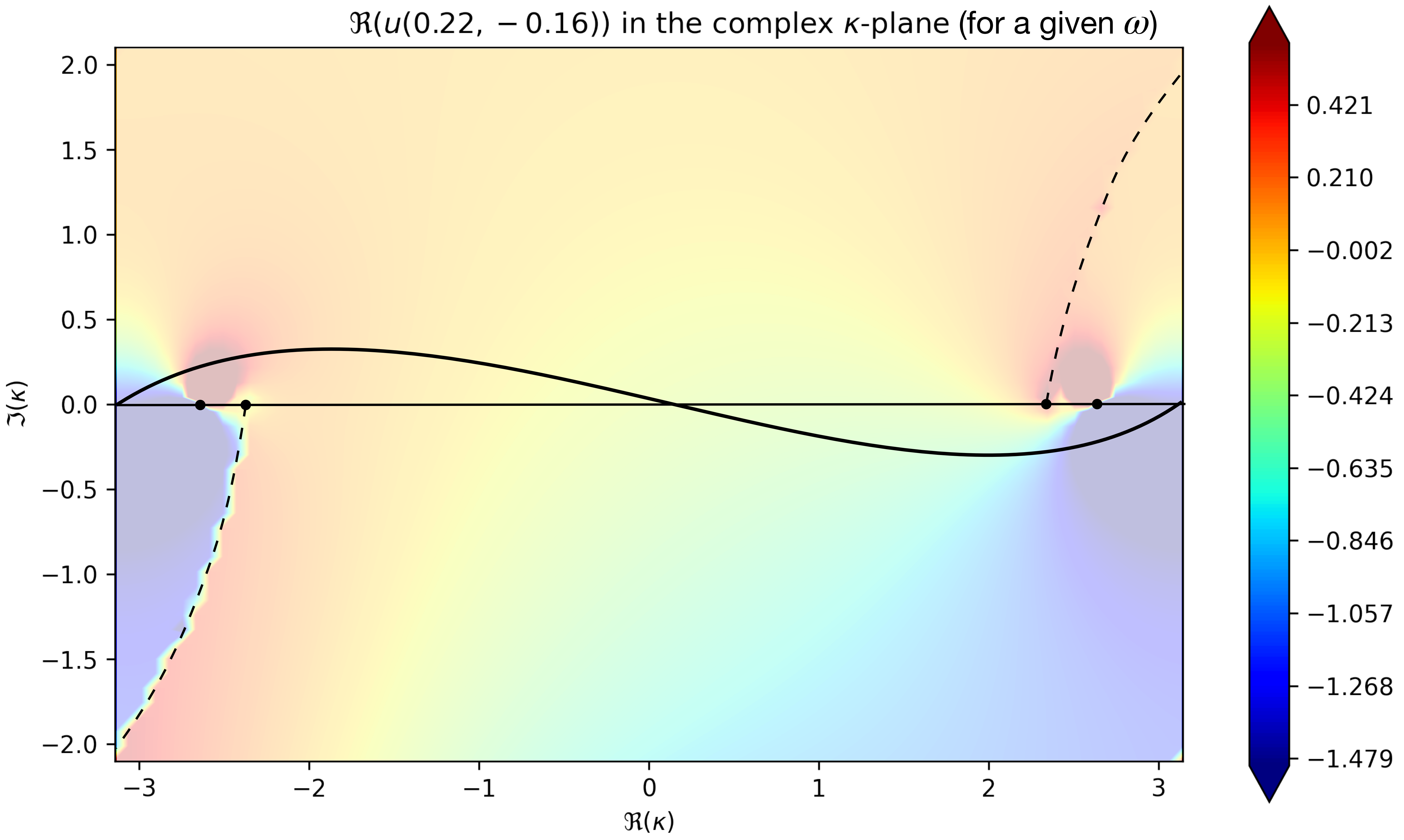

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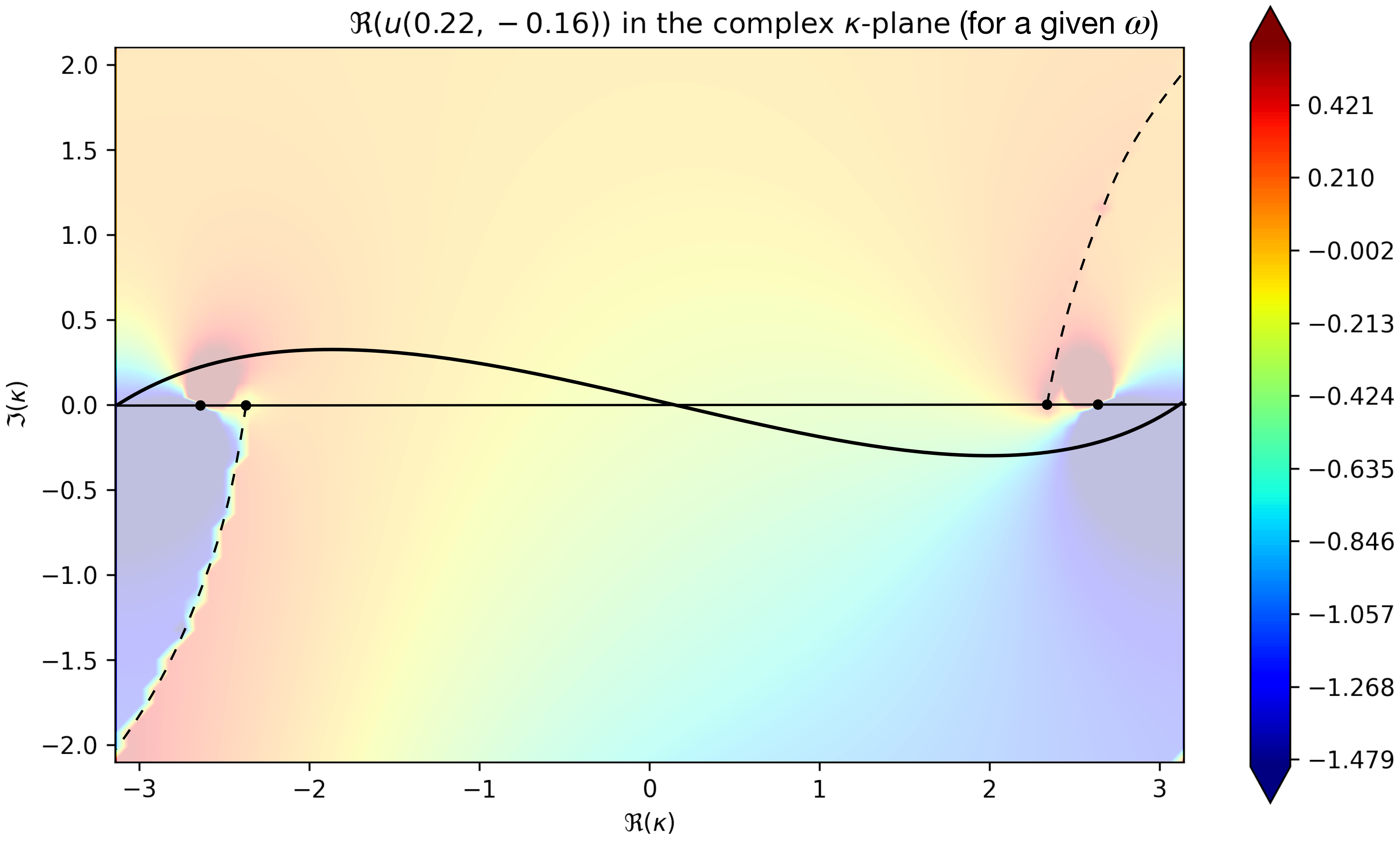

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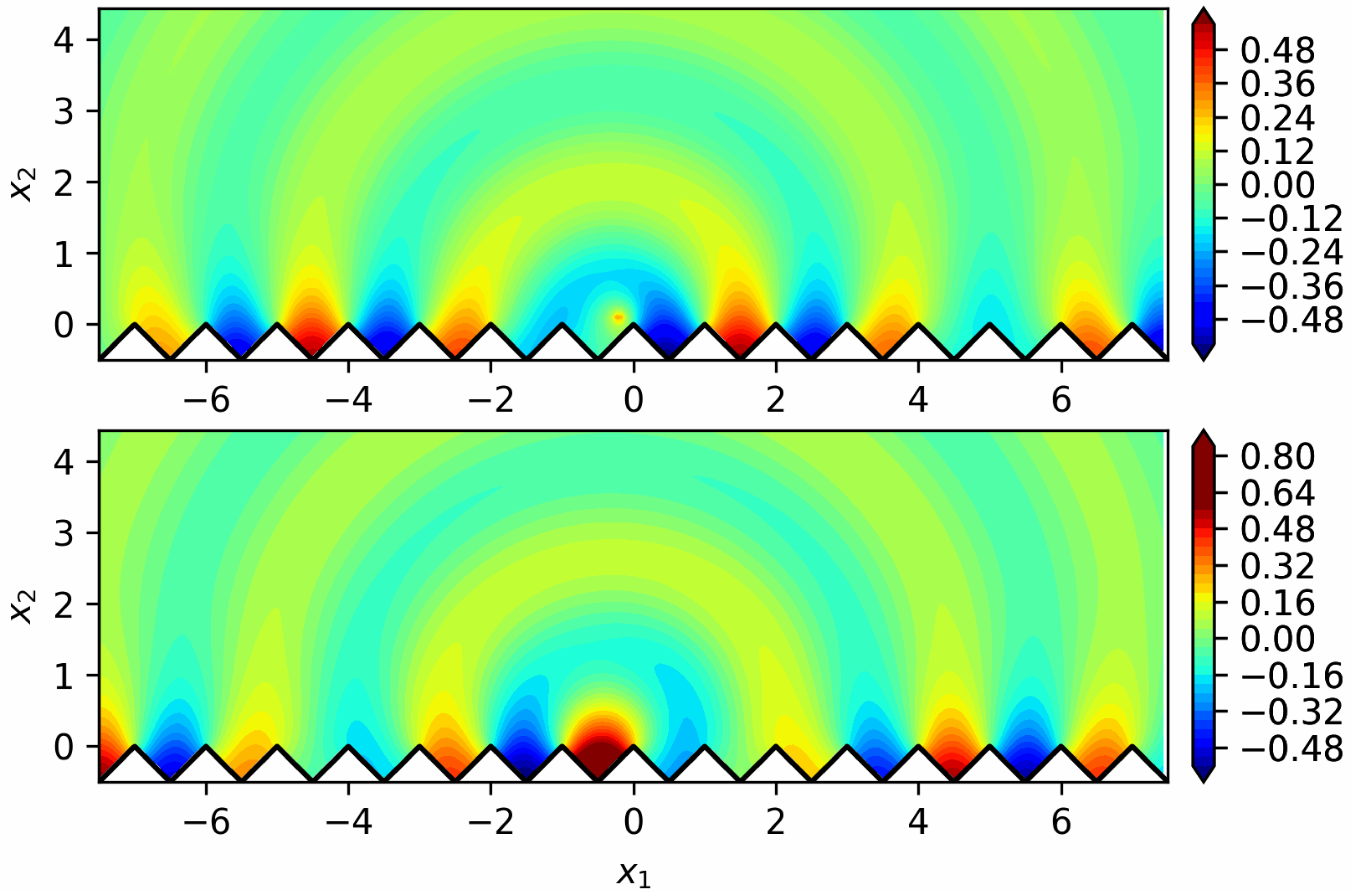
Time-propagation of the total field away from the source (for a single ω)



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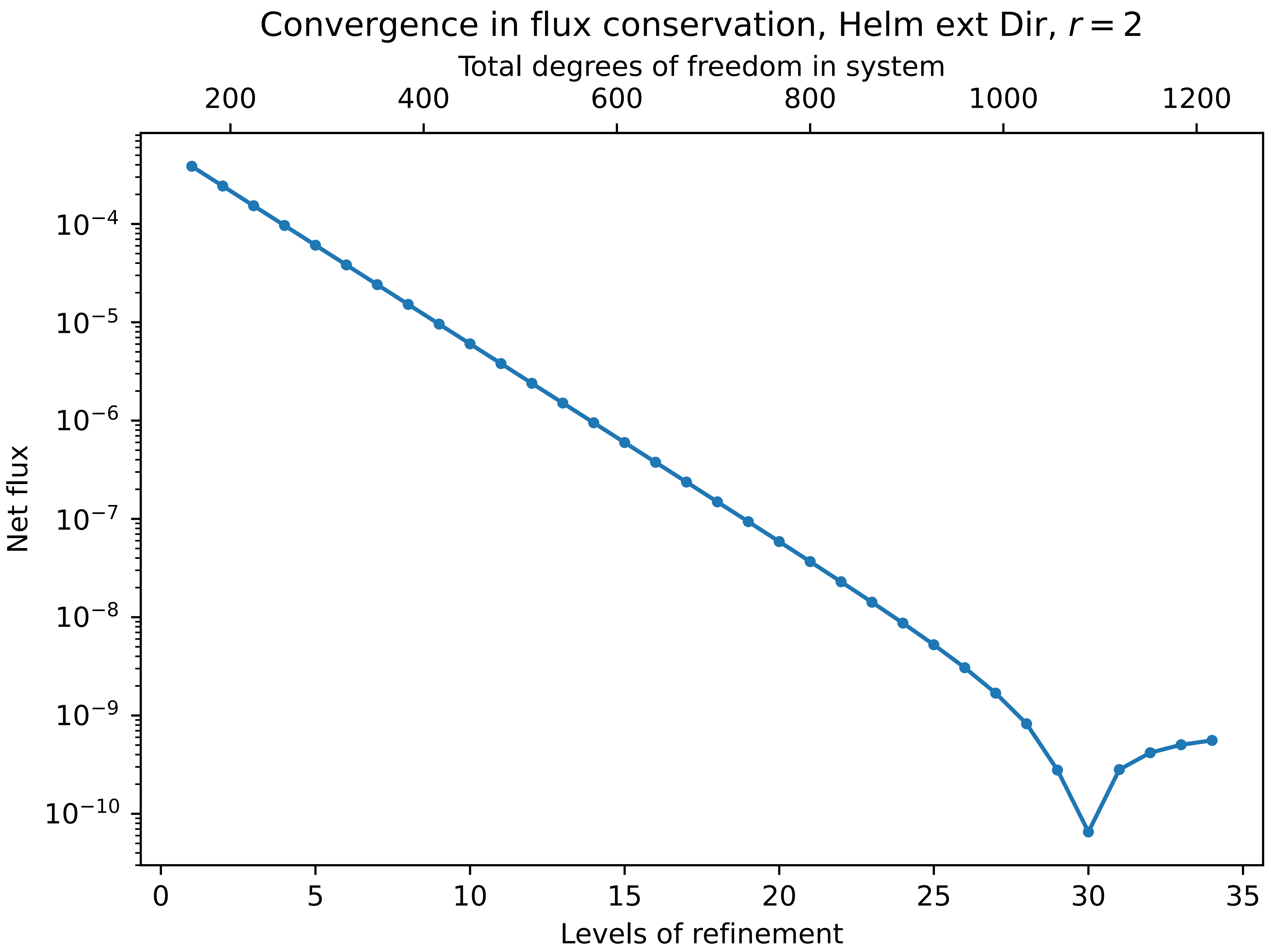

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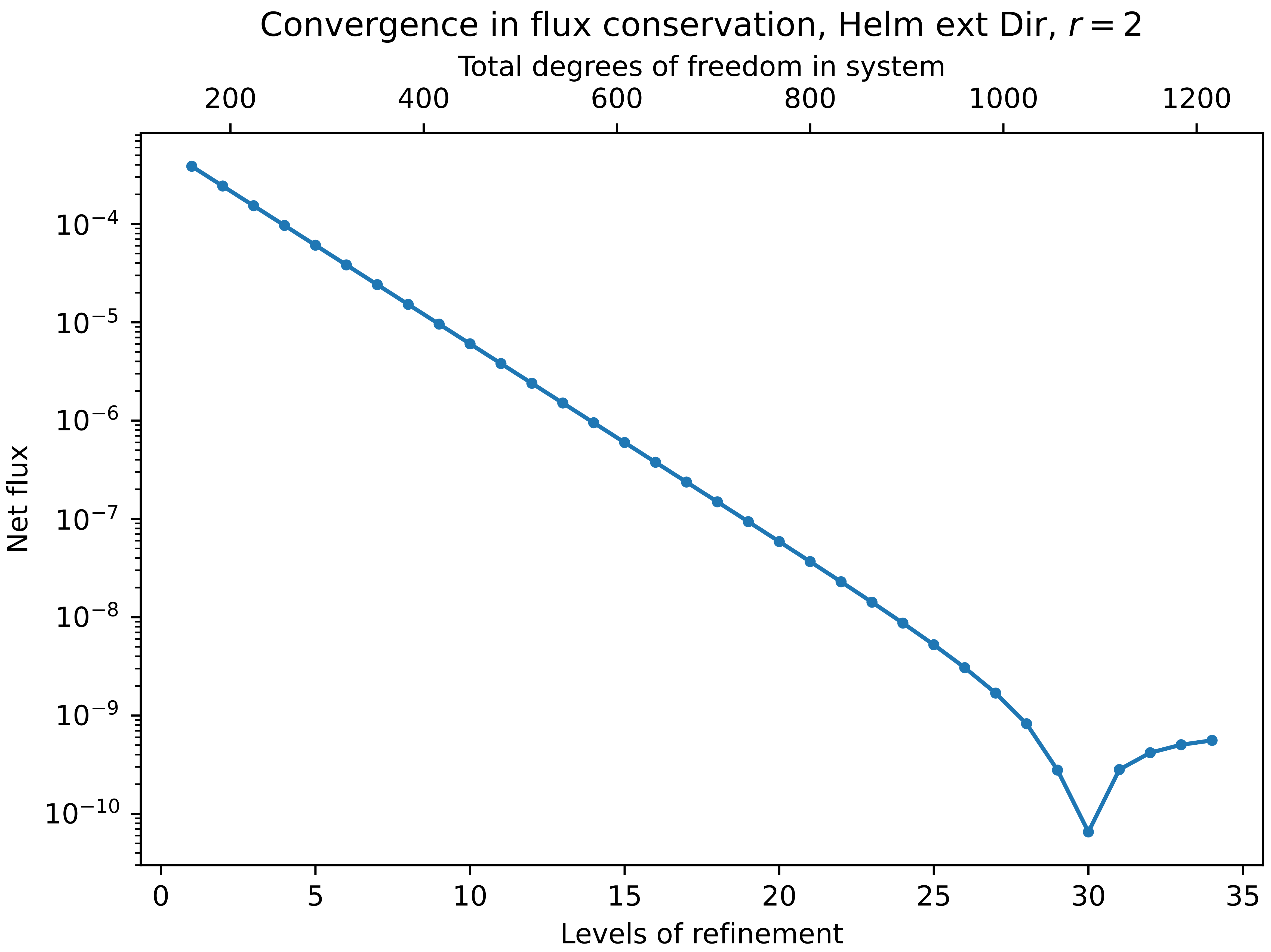

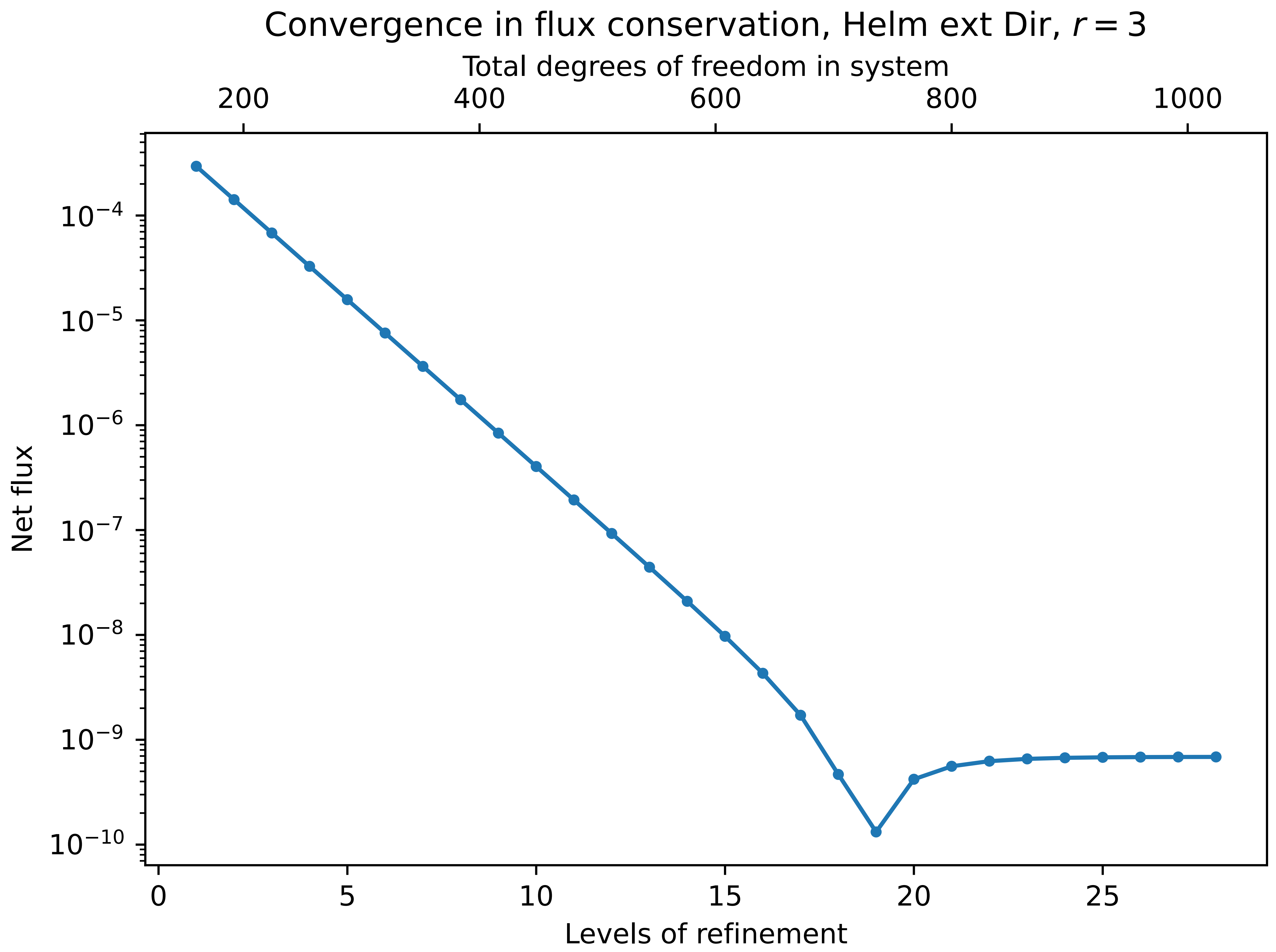

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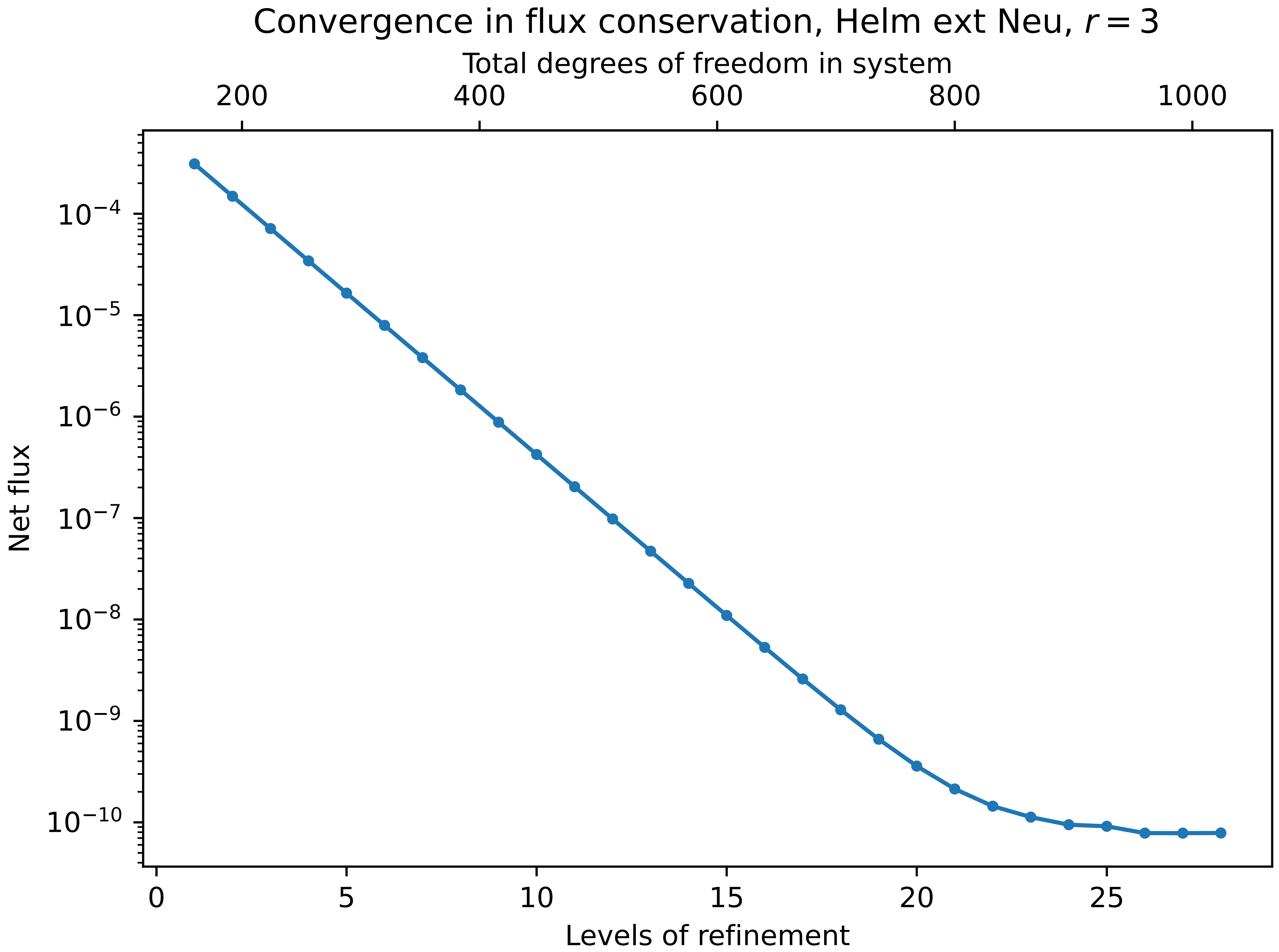

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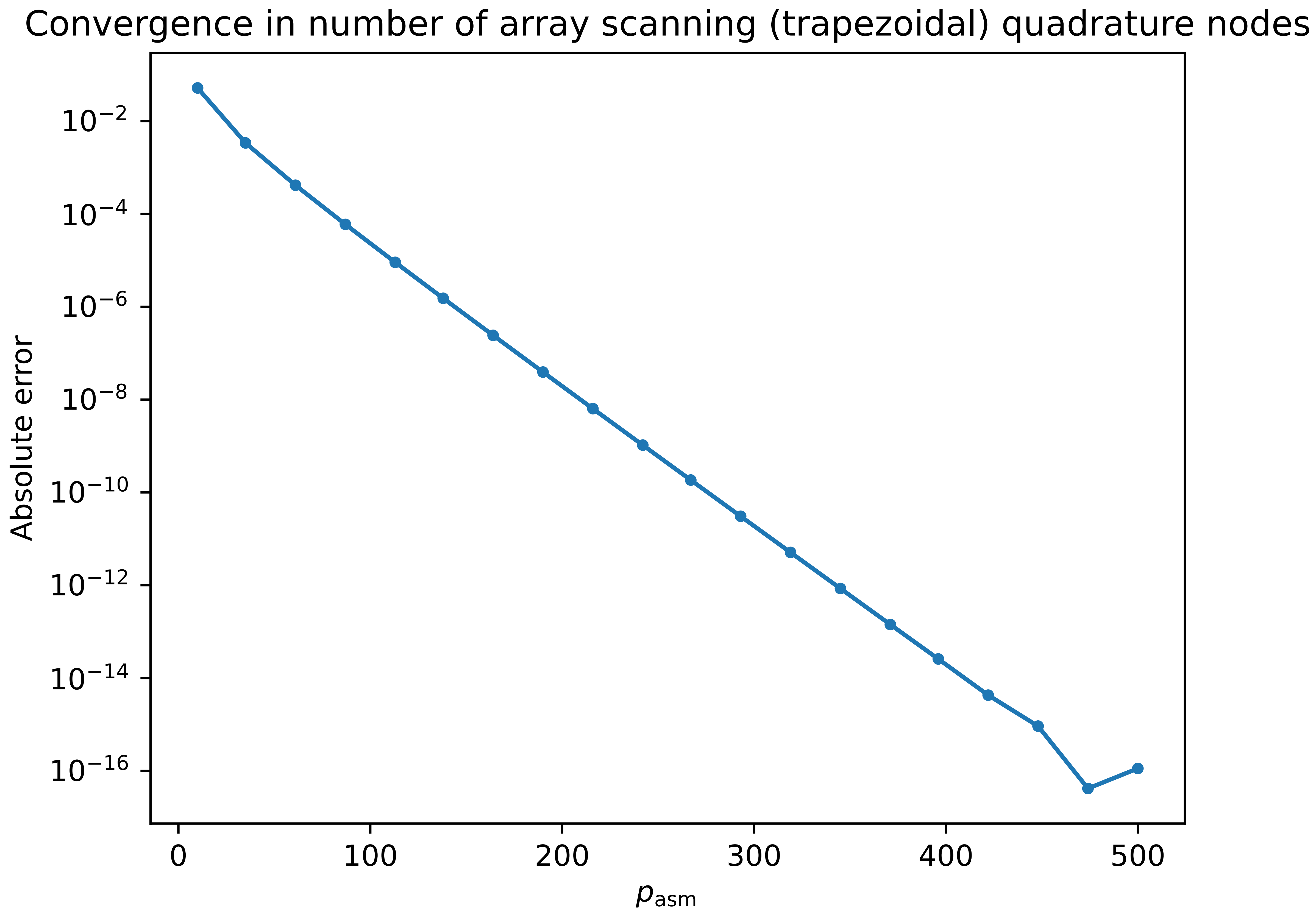

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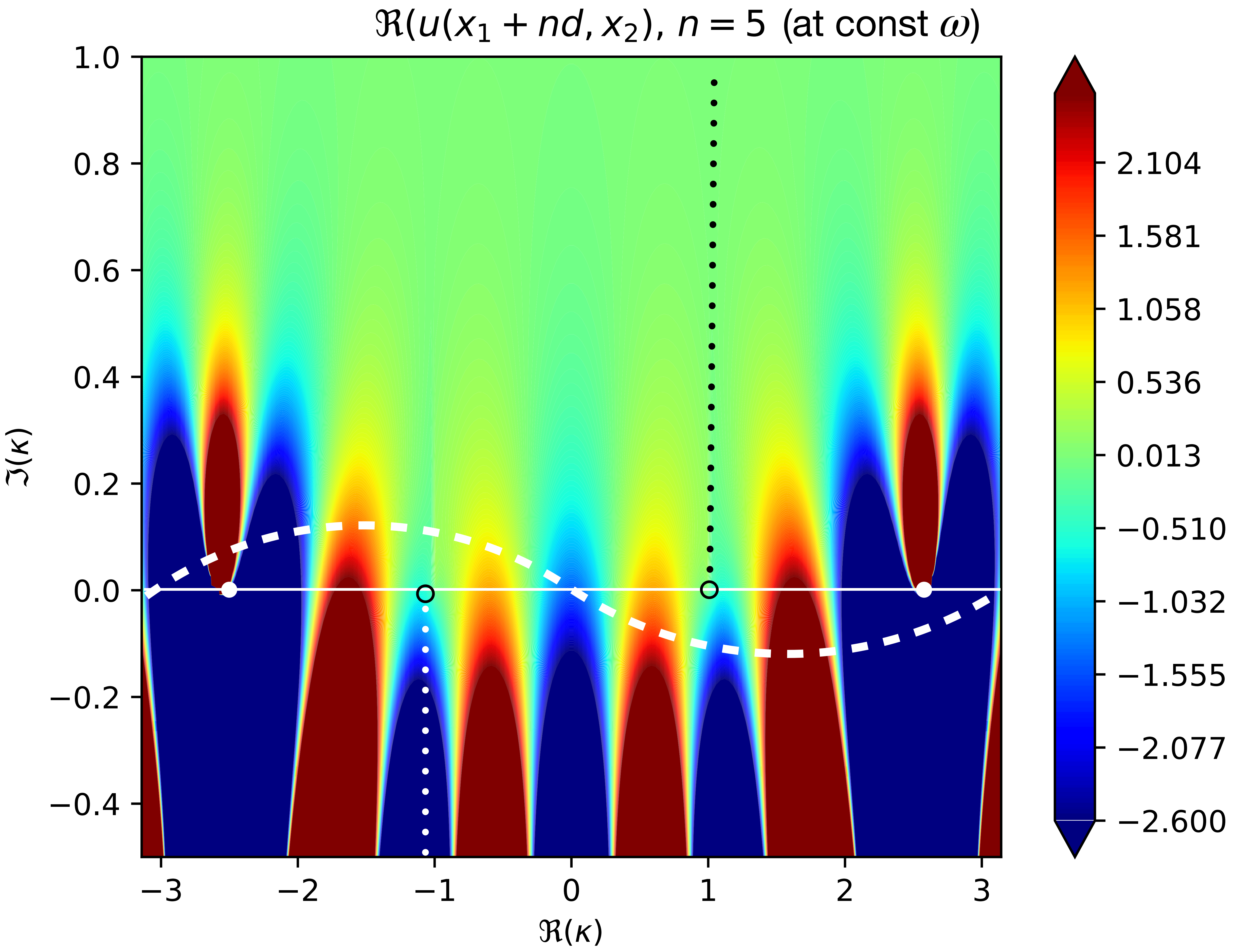

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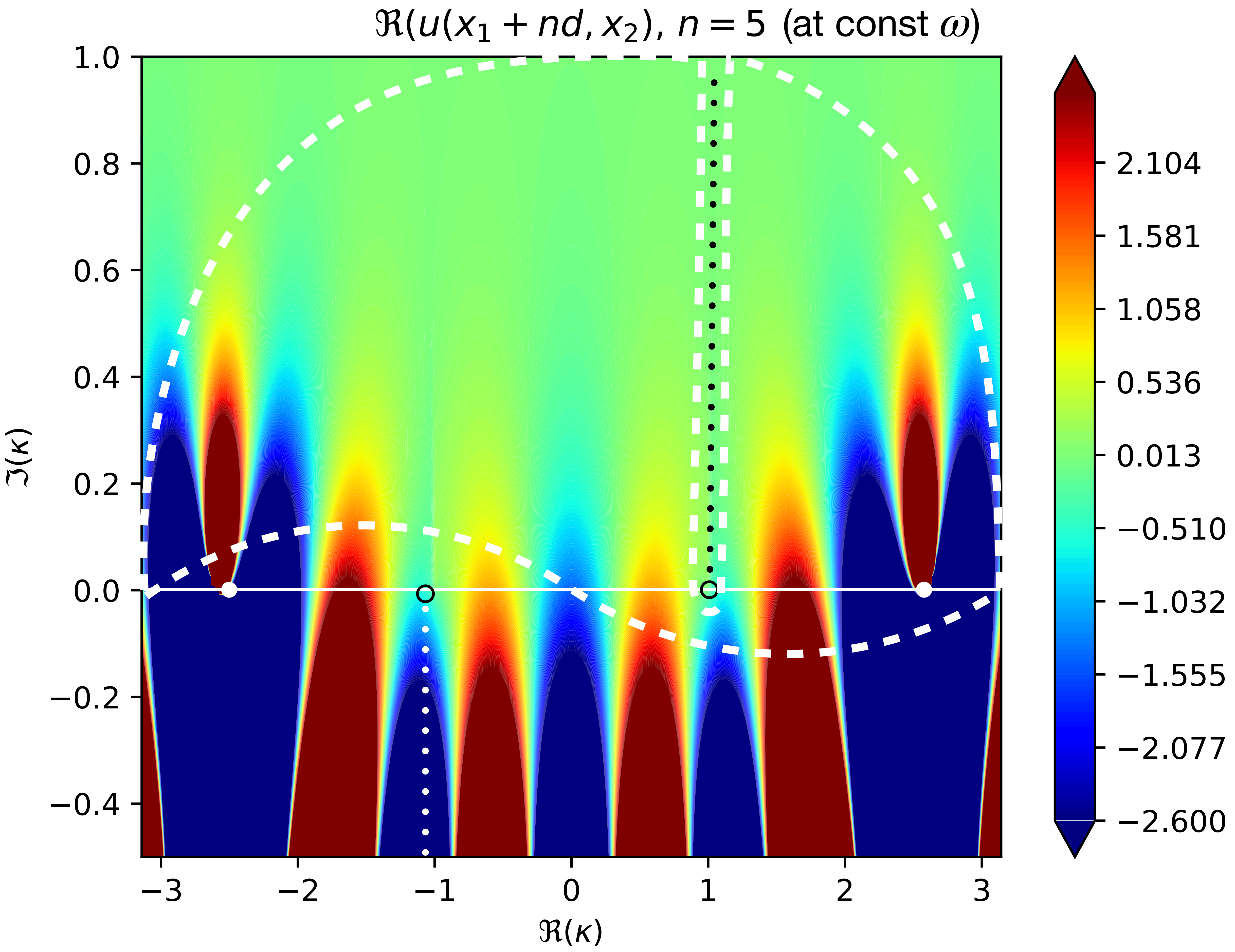

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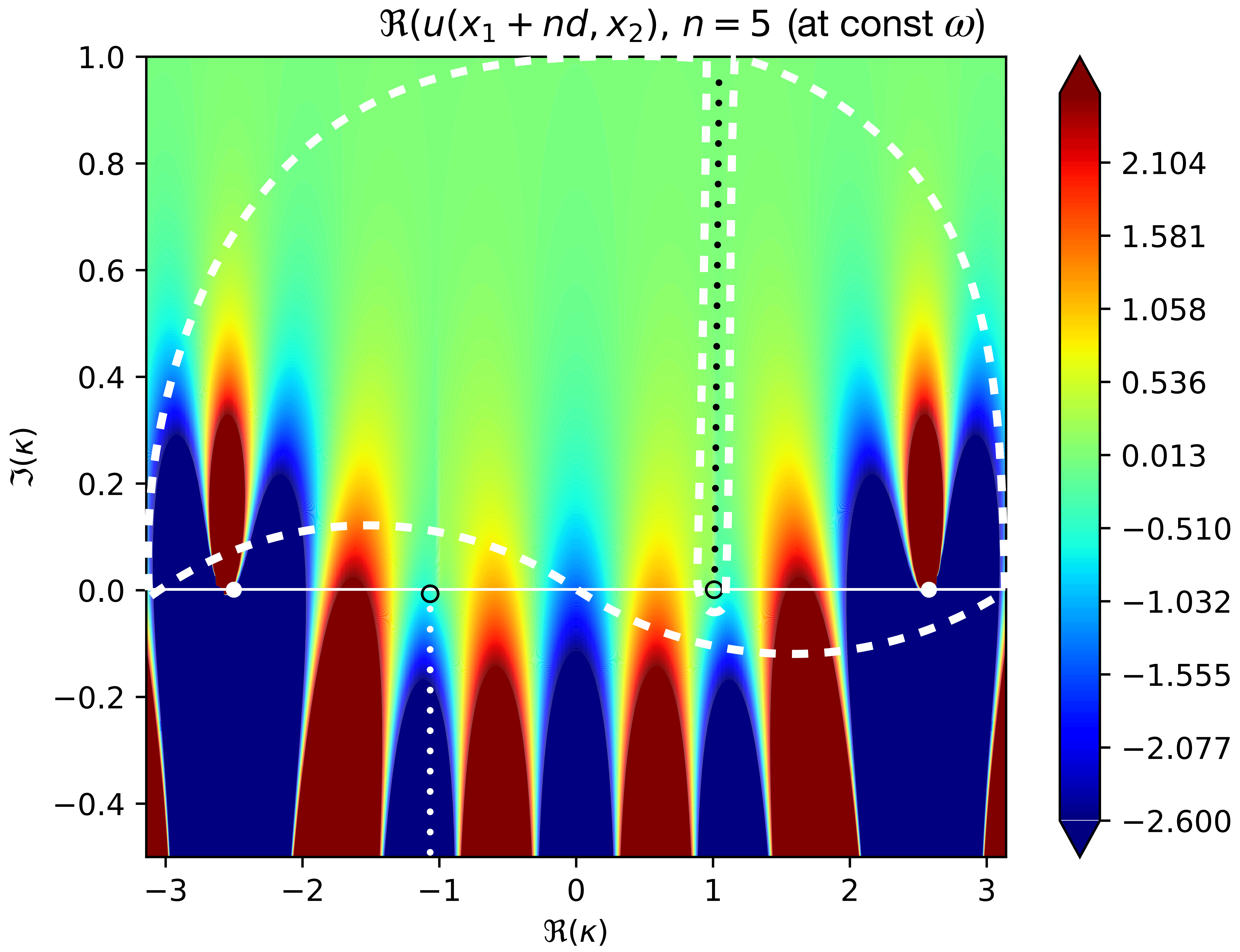

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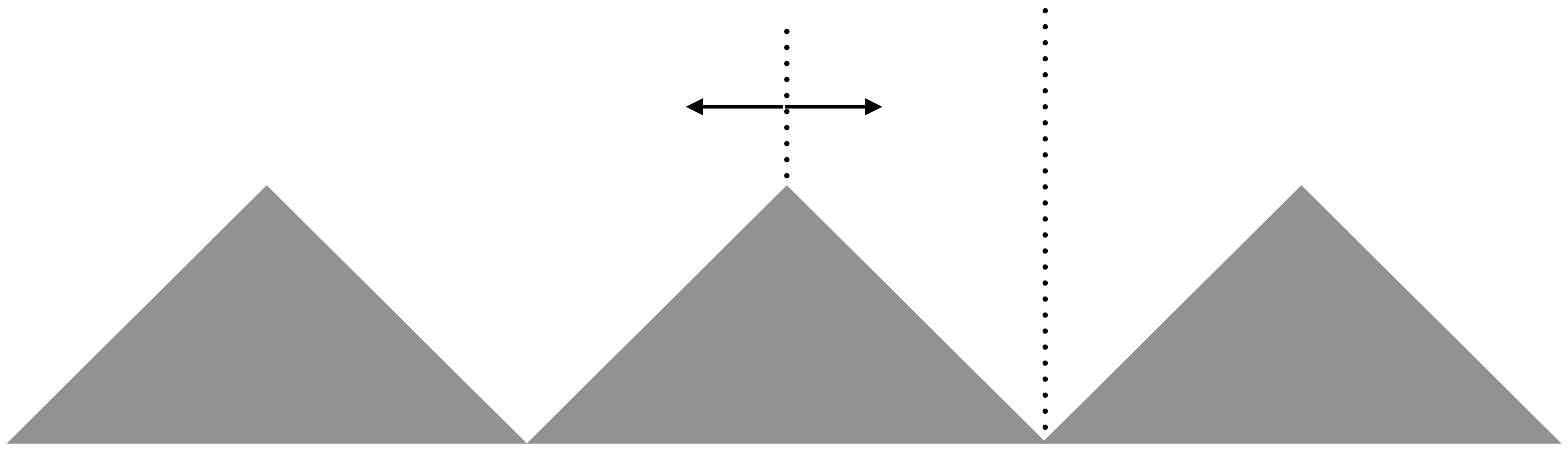

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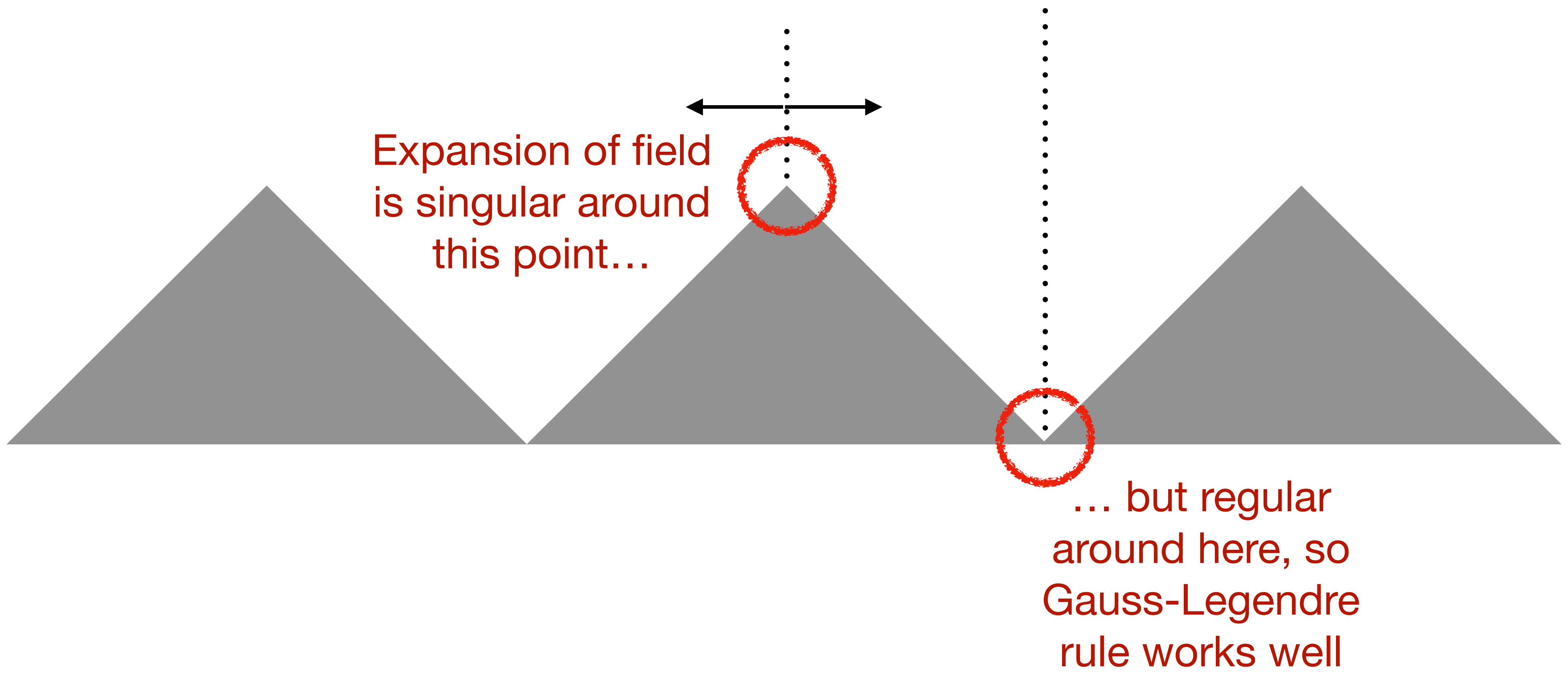

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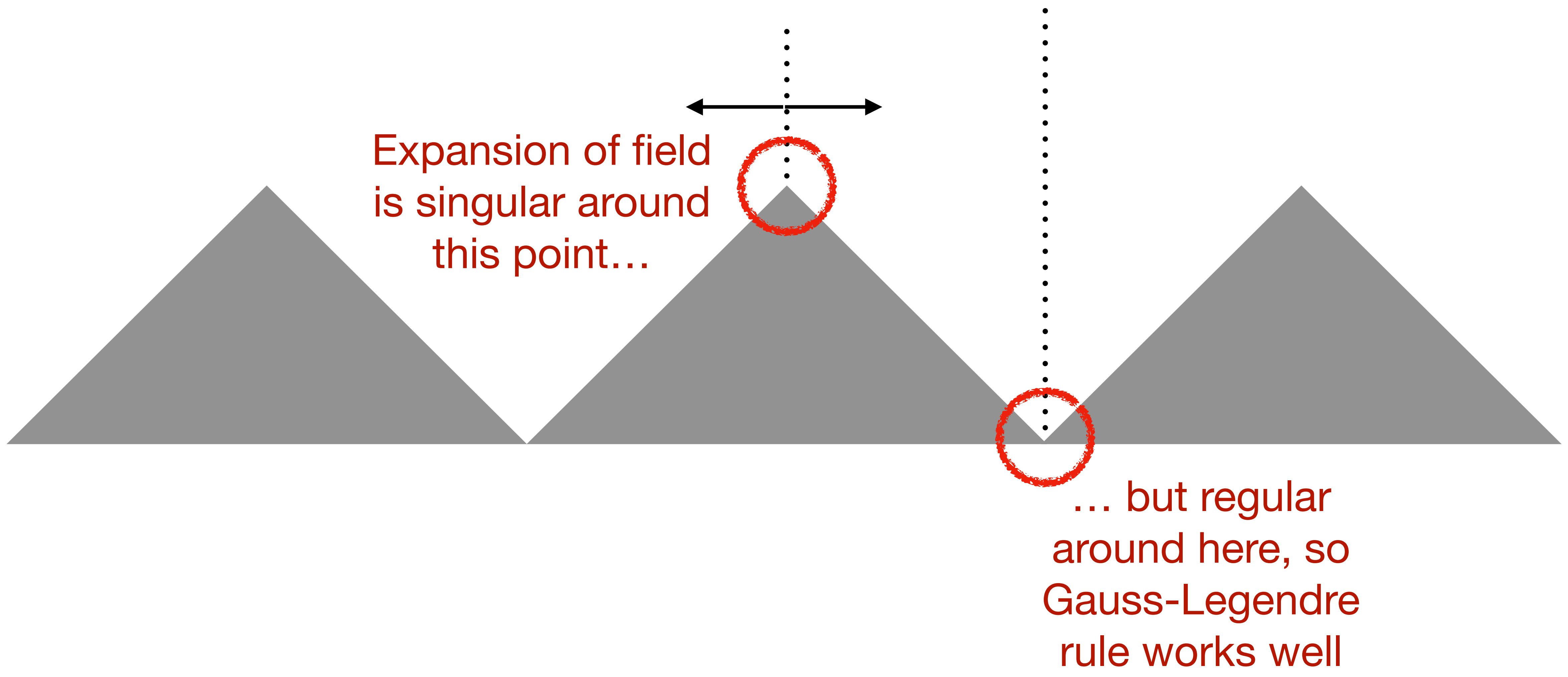

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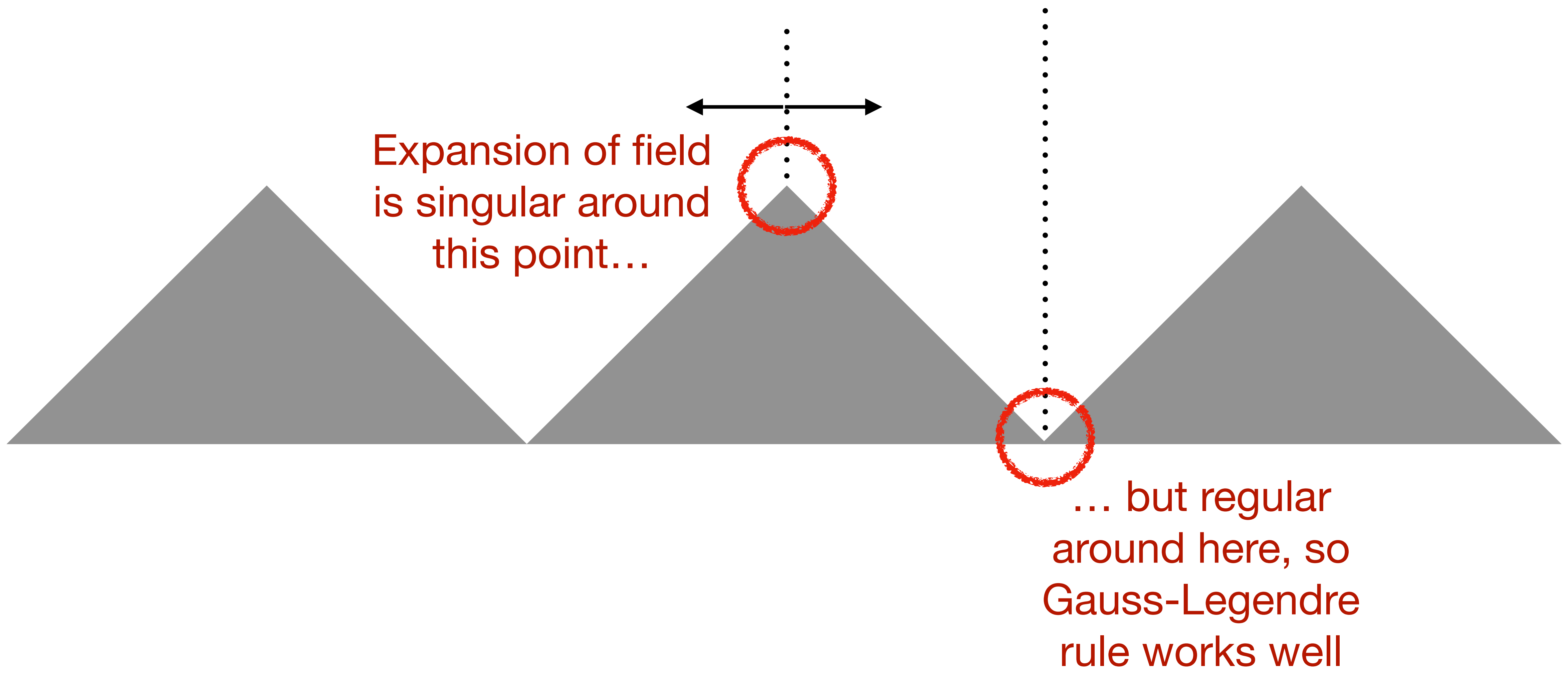

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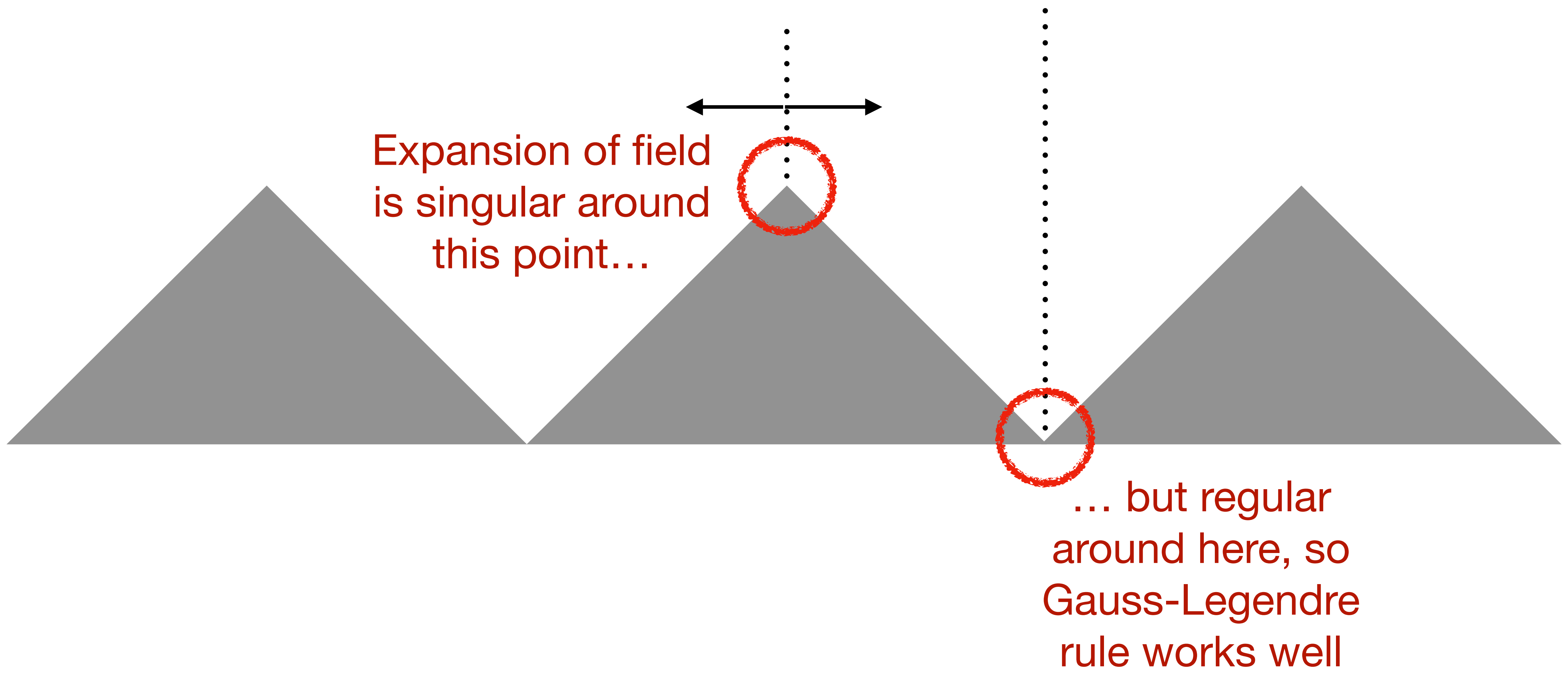

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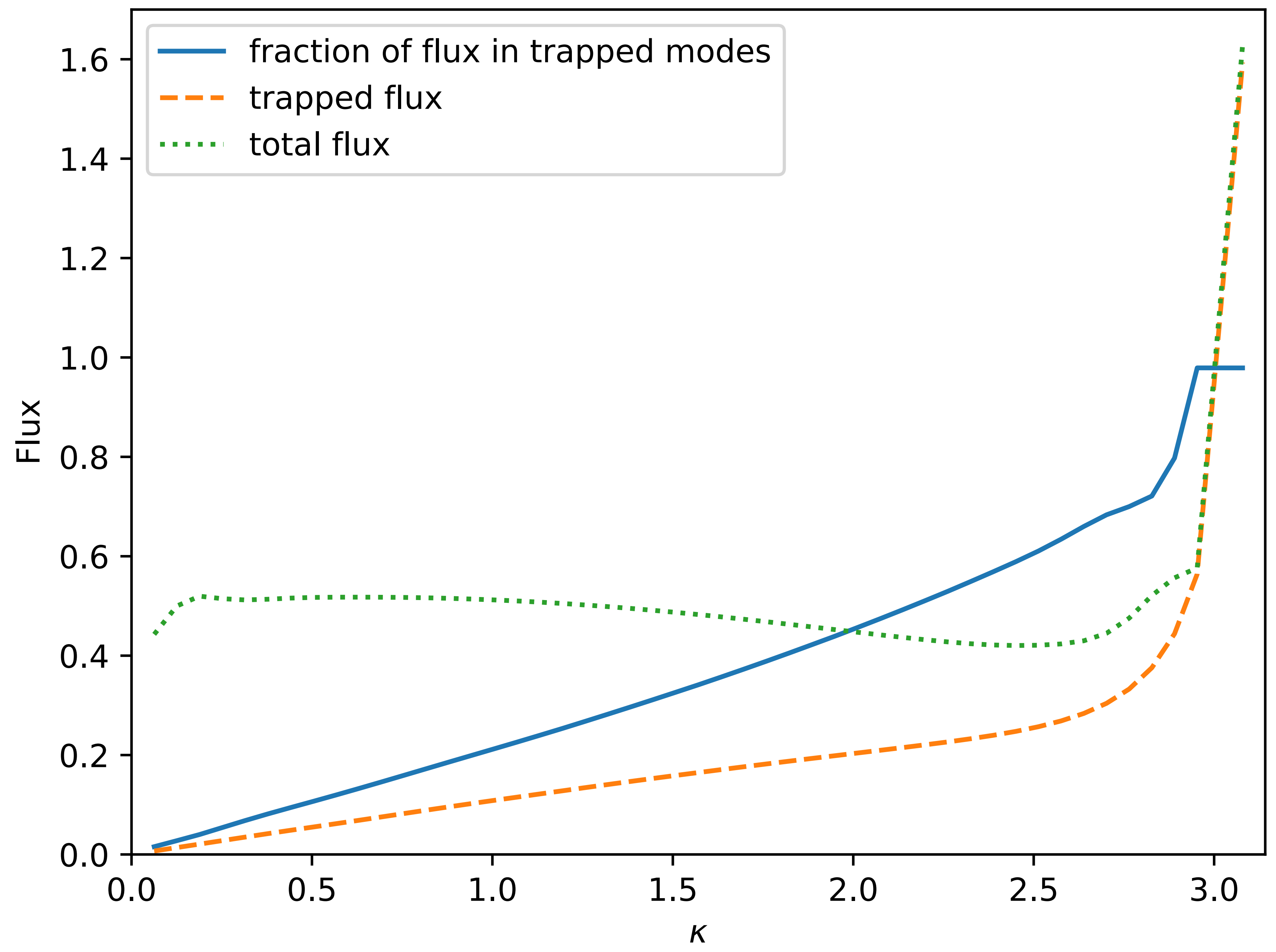

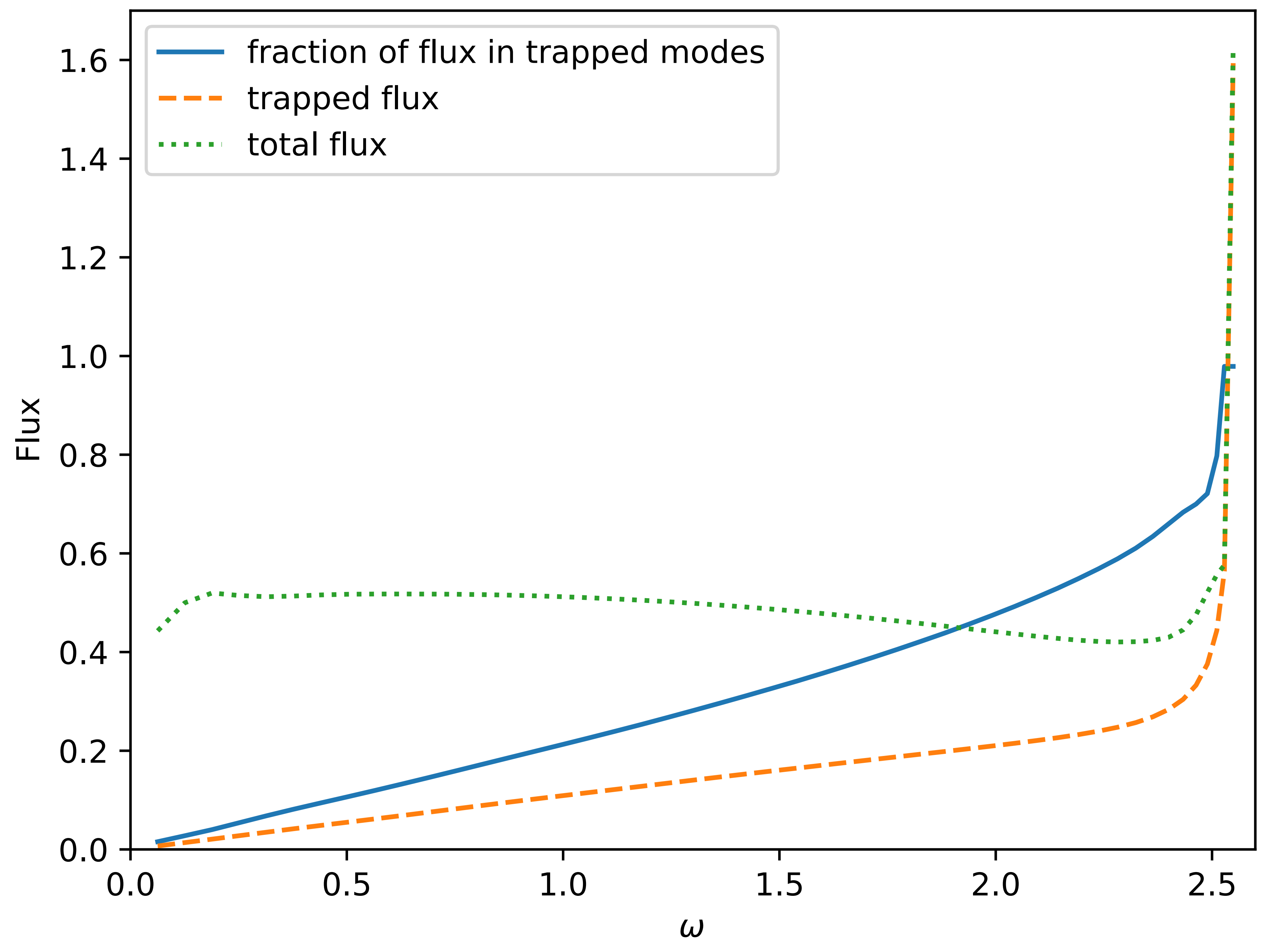

Future work

- How does the positioning of the source affect the power in trapped modes?

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- Can we do this in 3D? Band structure is more complex.

Thank you!